

PUBBLICAZIONI
DEL
COMITATO NAZIONALE PER LE MANIFESTAZIONI CELEBRATIVE
DEL IV CENTENARIO DELLA NASCITA DI GALILEO GALILEI

VOLUME II

TOMO 1

ATTI
DEL
CONVEGNO SULLA RELATIVITÀ GENERALE
PROBLEMI DI ENERGIA
E ONDE GRAVITAZIONALI

PROCEEDINGS OF THE MEETING ON GENERAL RELATIVITY
PROBLEMS OF ENERGY AND GRAVITATIONAL WAVES



FIRENZE
G. BARBÈRA EDITORE

1965

IV CENTENARIO DELLA NASCITA
DI GALILEO GALILEI
1564-1964

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PER LE MANIFESTAZIONI CELEBRATIVE

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Celebrazioni

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VOLUME IV
Testimonianze Galileiane

C. BARBÈRA, EDITORE
FIRENZE

**IV CENTENARIO DELLA NASCITA
DI
GALILEO GALILEI
1564 - 1964**

Il ritratto di Galileo riprodotto nella pagina seguente, è tratto da:
J. DE SANDRART: Academia nobilissimae artis pictoriae,
Noribergae, 1683.



*J. de Sandrart delineavit
B. Kilian sculpsit*

IV CENTENARIO DELLA NASCITA DI GALILEO GALILEI
1564 - 1964

MANIFESTAZIONI CELEBRATIVE
SOTTO
L'ALTO PATRONATO DEL PRESIDENTE DELLA REPUBBLICA

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IV CENTENARIO DELLA NASCITA DI GALILEO GALILEI
1564 - 1964

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1965

PROSPETTO GENERALE
DELLE
PUBBLICAZIONI

VOLUME I **Celebrazioni**

VOLUME II **Atti dei Convegni**

TOMO 1 *Atti del Convegno sulla relatività generale: problemi di energia
e onde gravitazionali*

*Proceedings of the Meeting on General Relativity: Problems of Energy and
Gravitational Waves*

Firenze, 9-12 Settembre 1964

TOMO 2 *Atti del Convegno sulle macchie solari*

Proceedings of the Meeting on Sunspots

Firenze, 9-12 Settembre 1964

TOMO 3 *Atti del Convegno sulla cosmologia*

Proceedings of the Meeting on Cosmology

Padova-Venezia, 14-16 Settembre 1964

TOMO 4 *Atti del Convegno sui campi magnetici solari e la spettroscopia
ad alta risoluzione*

*Proceedings of the Meeting on Solar Magnetic Fields and High Resolution
Spectroscopy*

Roma, 14-16 Settembre 1964

TOMO 5 *Atti del Simposio su Galileo Galilei nella storia e nella filosofia
della scienza*

*Proceedings of the Symposium on Galileo Galilei in History and Philosophy
of Science*

Firenze-Pisa, 14-16 Settembre 1964

TOMO 6 *Atti del Convegno sulla filosofia naturale, oggi*

Proceedings of the Meeting on Natural Philosophy To-day

Pisa, 17-21 Settembre 1964

VOLUME III
IN TRE TOMI **Saggi su Galileo Galilei**

VOLUME IV **Testimonianze galileiane**

A CURA DEL

COMITATO NAZIONALE PER LE MANIFESTAZIONI CELEBRATIVE

VOLUME II

ATTI DEI CONVEGNI

SOTTOCOMITATO PER L'ORGANIZZAZIONE DEI CONVEGNI

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VASCO RONCHI - LEONIDA ROSINO - GIORGIO SESTINI

GIULIANO TORALDO DI FRANCIA

ATTI DEI CONVEGNI

TOMO 1

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Firenze
Istituto Matematico « Ulisse Dini »
9-12 Settembre 1964

COMITATO ESECUTIVO FIORENTINO

CARLO CATTANEO - GIOVANNI GODOLI - DEMORE QUILGHINI

RAFFAELLO RAMAT - GUCLIELMO RIGHINI - GIORGIO SESTINI

GIULIANO TORALDO DI FRANCIA

PRESENTAZIONE

L'idea di un convegno internazionale sulla relatività generale, nel quadro delle celebrazioni per il IV Centenario della nascita di Galileo, sorse nell'estate del 1962 in seno al Comitato scientifico nominato dal Sindaco di Firenze al fine di studiare e proporre manifestazioni idonee a tali celebrazioni.

Il Comitato, composto da G. Righini, G. Sestini, G. Toraldo dell'Università di Firenze, fu unanime nel ritenere che una maniera degna di onorare Galileo fosse quella di indire a Firenze alcuni convegni internazionali ad alto livello, e suggerì al Sindaco di promuovere a Firenze, fra altre manifestazioni, una riunione del COSPAR, un convegno sulla relatività generale ed uno sulle macchie solari, in quanto relativi a campi di ricerca che negli studi di Galileo trovano iniziale impulso di decisiva importanza. Tali convegni, per l'attualità dei problemi in discussione, sembravano poi particolarmente significativi a celebrare la fecondità delle idee di Galileo e sopra tutto del metodo galileiano.

Da questa prima idea fiorentina nacquero, con ben maggior risonanza internazionale e varietà di temi in discussione, le « Giornate galileiane », promosse dal Comitato nazionale per le celebrazioni galileiane ed organizzate a Firenze, Padova, Roma e Pisa.

Nel quadro delle « Giornate », il Convegno sulla relatività generale, per il tema: problemi di energia e onde gravitazionali fu articolato, sullo schema adottato per tutti i Convegni delle « Giornate », in sedute destinate alla discussione di temi presentati da relatori espressamente invitati dal Comitato organizzatore, composta da C. Cattaneo dell'Università di Roma e G. Righini, G. Sestini, G. Toraldo, D. Quilghini dell'Università di Firenze. Anche la partecipazione al Convegno fu esclusivamente per invito e alla discussione fu riservato un tempo uguale a quello concesso al relatore per l'esposizione del tema.

Il cospicuo contributo stanziato dal Comune di Firenze e la generosa collaborazione dell'Azienda Autonoma di Turismo di Firenze, hanno consentito

al Comitato organizzatore di offrire ai partecipanti al Convegno, oltre al ricevimento del Sindaco, Giorgio La Pira, in occasione dell'apertura delle « Giornate », alcune manifestazioni di contorno atte ad allietare il soggiorno fiorentino dei graditi ospiti.

Al Sindaco di Firenze, all'Assessore alla Cultura e belle arti, R. Ramat, e al Presidente dell'Azienda di Turismo va il più vivo e riconoscente ringraziamento del Comitato organizzatore.

Ai sedici relatori — P. G. Bergmann, H. Bondi, W. B. Bonnor, R. Debever, B. Finzi, V. A. Fock, J. Géheniau, D. Ivanenko, A. Lichnerowicz, A. Mercier, C. Møller, N. Rosen, A. Schild, A. H. Taub, M. A. Tonnelat e H. J. Treder —, ai presidenti delle sette sedute — P. G. Bergmann, J. Wheeler, C. Møller, V. A. Fock, N. Rosen, A. G. Walker e A. Lichnerowicz —, ai segretari delle sedute — I. Gasparini Cattaneo, V. Cantoni e E. Tonti —, a quanti con la partecipazione alla discussione hanno collaborato alla riuscita del Convegno, va il più cordiale “grazie!” del Comitato organizzatore, estendibile a quanti accettarono l'invito a partecipare al Convegno.

Un particolare ringraziamento a tutto il personale dell'Istituto Matematico « U. Dini » dell'Università di Firenze, sede del Convegno.

Le discussioni vaste ed animate, che questi Atti riassumono nelle linee generali, costituiscono il consuntivo del Convegno sulla cui utilità tutti i partecipanti hanno convenuto, anche per l'opportunità di incontrarsi e discutere, offerta, nel nome di Galileo, a settantotto studiosi di dodici Paesi.

Il Segretario del Convegno

GIORGIO SESTINI

dell'Università di Firenze

DIARIO DEL CONVEGNO

Mercoledì, 9 Settembre 1964

Mattino

ORE 9 - FIRENZE, ISTITUTO MATEMATICO DELL'UNIVERSITÀ

Presidente: P. G. BERGMANN

Segretari: I. GASPARINI CATTANEO e V. CANTONI

C. MØLLER: *Energy and momentum carried by gravitational waves.**

M. A. TONNELAT: *Définition et rôle d'une énergie gravitationnelle dans les théories minkowskiennes du champ de gravitation.**

ORE 12 - FIRENZE, BIBLIOTECA NAZIONALE CENTRALE

Inaugurazione della Mostra di documenti e cimeli galileiani.

Pomeriggio

ORE 16 - FIRENZE, ISTITUTO MATEMATICO DELL'UNIVERSITÀ

Presidente: J. WHEELER

Segretari: I. GASPARINI CATTANEO e V. CANTONI

N. ROSEN: *Energy and gravitational waves in bi-metric relativity theory.**

J. GÉHÉNIAU: *Sur le formalisme hamiltonien en relativité générale.**

Giovedì, 10 Settembre 1964

Mattino

ORE 8.30 - FIRENZE, ISTITUTO MATEMATICO DELL'UNIVERSITÀ

Presidente: C. MØLLER

Segretari: I. GASPARINI CATTANEO e V. CANTONI

V. A. FOCK: *Les principes mécaniques de Galilée et la théorie d'Einstein.**

* Relazione.

A. H. TAUB: *The motion of multipoles in general relativity.**

W. B. BONNOR: *Approximate methods and gravitational radiation.**

Pomeriggio

ORE 16 - FIRENZE, ISTITUTO MATEMATICO DELL'UNIVERSITÀ

Presidente: V. A. FOCK

Segretari: I. GASPARINI CATTANEO e V. CANTONI

H. BONDI: *Gravitational waves.**

ORE 17.30 - FIRENZE, CHIESA DI SANTA CROCE

Omaggio alla tomba di Galileo.

ORE 19 - FIRENZE, FORTE DI BELVEDERE

Cocktail offerto dall'Azienda Autonoma di Turismo di Firenze.

Venerdì, 11 Settembre 1964

Mattino

ORE 8.30 - FIRENZE, ISTITUTO MATEMATICO DELL'UNIVERSITÀ

Presidente: N. ROSEN

Segretari: I. GASPARINI CATTANEO e V. CANTONI

A. LICHNEROWICZ: *Champ de Dirac, champ du neutrino et transformations $\mathcal{C}$, $\mathcal{P}$, $\mathcal{I}$ sur un espace-temps courbe.**

A. MERCIER: *Les théorèmes de conservation et l'idée d'une théorie unitaire.**

R. DEBEVER: *Local tetrad in general relativity.**

Pomeriggio

ORE 16 - FIRENZE, ISTITUTO MATEMATICO DELL'UNIVERSITÀ

Presidente: A. G. WALKER

Segretari: I. GASPARINI CATTANEO ed E. TONTI

D. IVANENKO: *Gravitation and unified picture of matter.**

R. P. KERR and A. SCHILD: *A new class of vacuum solutions of the Einstein field equations.**

Sera

ORE 21.30 - FIRENZE, PALAZZO VECCHIO, SALONE DEI DUGENTO

Concerto, offerto dal Comune di Firenze, di musiche del Rinascimento italiano eseguite dal Complesso fiorentino di musica antica.

* Relazione.

Sabato, 12 Settembre 1964

Mattino

ORE 8.30 - FIRENZE, ISTITUTO MATEMATICO DELL'UNIVERSITÀ

Presidente: A. LICHTNEROWICZ

Segretari: I. GASPARINI CATTANEO ed E. TONTI

P. G. BERGMANN: *Radiation and observables*.*

H. J. TREEDER: *The gravitons and Einstein's particle problem*.*

CH. GOILLOT et O. COSTA DE BEAUREGARD: *Vérification expérimentale de la théorie de l'effet inertial de spin (E. I. S.)*.**

S. DESER: *Problems and prospects in quantization of relativity*.**

M. A. TONNELAT: *Parole di commiato a conclusione del Convegno*.

Pomeriggio

ORE 15.30 - VINCI

Visita ai luoghi leonardiani e merenda toscana.

* Relazione.

** Comunicazione.

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Relativity from Galileo to Einstein. (*)

B. FINZI

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1. - Galilean relativity.

The principle of relativity is introduced by Galileo in his letter to Ingoli, reproduced in the *Dialogo sopra i due massimi sistemi del mondo*, by means of a well known mental experiment. Galileo says:

« Biserratevi con qualche amico nella maggiore stanza che sia sotto coverta di alcun gran navilio, e quivi fate d'aver mosche, farfalle e simili animalletti volanti; siavi anco un gran vaso d'acqua, e dentrovi de' pescetti; suspendasi anco in alto qualche secchiello, che a goccia a goccia vadia versando dell'acqua in un altro vaso di angusta bocca, che sia posto a basso: e stando ferma la nave, osservate diligentemente come quelli animalletti volanti con pari velocità vanno verso tutte le parti della stanza; i pesci si vedranno andar notando indifferentemente per tutti i versi; le stille cadenti entreranno tutte nel vaso sottoposto; e voi, gettando all'amico alcuna cosa, non più gagliardamente la dovrete gettare verso quella parte che verso questa, quando le lontananze sieno eguali; e saltando voi, come si dice, a piè giunti, eguali spazii passerete verso tutte le parti. Osservate che avrete diligentemente tutte queste cose, benché niun dubbio ci sia che mentre il vassello sta fermo non debbano succeder così, fate muover la nave con qualsivoglia velocità; ché, (pur che il moto sia uniforme e non fluttuante in qua e in là) voi non riconoscerete una minima mutazione in tutti li nominati effetti, né da alcuno di quelli potrete comprender se la nave cammina o pure sta ferma: voi saltando passerete nel tavolato i medesimi spazii che prima, né, perché la nave si muova velocissimamente, farete maggior salti verso la poppa che verso la prua, benché nel tempo che voi state in aria, il tavolato sottopostovi scorra verso la parte contraria al vostro salto; ... le goccioline cadranno come prima nel vaso inferiore senza caderne pur una verso poppa ...; i pesci nella lor acqua non con più fatica

(*) Translation of the opening lecture of the *Galilean Days* and of the *Symposium on the Relativity* (Florence, Palazzo Vecchio, 8th September, 1964).

noteranno verso la precedente che verso la susseguente parte del vaso ...; e finalmente le farfalle e le mosche continueranno i loro voli indifferentemente verso tutte le parti ... ».

Galileo's principle of relativity therefore states that, passing from one observer to another, moving with a translatory rectilinear uniform motion relative to the former, all mechanical phenomena remain unaltered, unaltered the laws governing them, unaltered all the intrinsic mechanical quantities experimentally checked. Mechanical laws must therefore remain unaltered, also in form, when the mathematical transformation that represents the passage from the temporal and spacial coordinates of one observer to those of the other is made. And if in particular these transformations leave unchanged the distance between any two point of the Euclidean geometric space, which, according to the postulates of classical kinematics, is considered as absolute, and leave unchanged all intervals of time that too considered as absolute, the transformations are extremely simple and are called « Galilean », although Galileo said nothing on this subject.

Galileo's principle of relativity is indeed more restrictive, but certainly more constructive than the simple « relativity of motion », principle this by which it only makes sense to speak of the motion of as body as regards another (rigid) which is the observer, a principle that had already made its appearance before Galileo, but which Galileo stated clearly and strongly, and was also clearly stated by Cartesius, Huygens and Newton, who, together with Galileo, can be called the founders of mechanics.

Galileo's principle of relativity, preposed to the « law of inertia » allows us to extend the first part, obvious and well known in ancient times, by which there exists an observer with respect to whom an isolated body « omne perseverare in statu suo quiescendi... », to the second part, by no means intuitive, in which the body, in the former conditions, « omne perseverare in statu suo movendi uniformiter in directum... ». In order to pass from the first to the second part of the law of inertia, it is sufficient to pass from the former observer, when with respect to it the body has a certain initial speed, to another observer, moving, with respect to the body, in uniform and rectilineal motion and at the same speed. Indeed relatively to this second mobile observer, the body is initially at rest and remains at rest for the first part of the law of inertia and for the principle of relativity.

In the same way Newton's « fundamental law » « Mutationem motus proportionalem esse vi motrici impressae... » is extended from the intuitive case of incipient motion, in which the initial speed is null, to the other, less intuitive, in which the speed is not null.

2. - Cartesian relativity.

Cartesius, in starting analytic geometry, was to implicitly state the invariability of the algebraic ratios that express geometrical laws, in the presence of a generic change of reference, that is a generic change of the directions of the axes of reference and of the position of their origin. Thus if three points are in line and the equation that ties the coordinates of the three points in a Cartesian reference shows that a certain expression is null, changing the reference the expression changes, but does not cease to be null, because it is a geometrical fact that the three points are in line independent by of the reference.

The Cartesian principle of relativity states the isotropy and the homogeneity of space. It concerns all physics because geometry is the first chapter of physics (conceptually the simplest) and because most physical laws are based on special concepts and representations.

Cartesian relativity is amplified with the introduction, by Gauss, of general spacial coordinates, and is even further amplified by operating in non Euclidean spaces with any number of dimensions, becoming « general geometric relativity », and here it finds its main algorithm in Ricci's tensor calculus. Nevertheless, even so generalized, this relativity concerns only the invariability of the laws when faced with the sole change of the geometric reference, and is less restrictive than Galileo's relativity, because it does not, like his, require invariability of form.

From geometry Cartesius was irresistibly led to extend the principle of general relativity to motion, so that it should include all physics, which according to him were only extension and motion. His opinion was supported by the fact that impact phenomena, in which he recognized the elementary mechanical facts, are invariable when in the presence of a generic continuous motion of the observer.

Huygens, who had so shrewdly exploited Galileo's relativity to establish the laws of impact, became convinced of the Cartesian thesis stating the principle of general relativity in physics, although Newton had rejected it. In fact Newton had observed that the reciprocity of the motion of bodies does not imply reciprocity of the connected mechanical phenomena: if a pail of water rotates uniformly around its vertical axis, relatively to the walls of the room it is in, the free surface of the water becomes paraboloidical, but this is not so if the pail remains still and the walls revolve round it at an opposite angular speed around the above-mentioned axis.

Nevertheless it was precisely the notion of reciprocity of motions that suggested to Newton the principle of action and reaction, which extends the reciprocity to the causes of motion, that is to forces.

3. - Classical relativistic mechanics.

In classical mechanics one passes from an inertial observer, for whom the law of inertia is valid, to another generically mobile as regards the first, for whom the law of inertia is not valid, without however varying either the duration or the mutual distances in the Euclidean space. The mechanics relative to this second observer are set up in the same way, according to Coriolis' theorem (which extends one of Clairaut) as those relative to an inertial observer, but adding to the effective forces the so-called « apparent forces »: of pull, such as for instance centrifugal force, and complementary ones, such as for instance compound centrifugal force which deviates towards the East all freely falling weights in reference to an earthly observer.

Subordinately to the rule (to the convention) just mentioned for the transformation of forces when one passes from an inertial observer to a non-inertial one, we may say also that classical relativistic mechanics satisfy a Cartesian principle of relativity, general may be, but submitted to Coriolis' rule (convention).

If on the other hand we think from the point of view of the so-called « thread school » of Reech and Andrade, who consider forces only those that can be substituted (at least in concept) by the tension of an elastic thread (measured from its stretch), Coriolis' rule can be replaced by the following of experimental nature: applying the traction of a thread to a particle, the acceleration, relative to a general reference frame, is increased by a quantity equal to the traction of the thread, divided by the mass.

In such a way the fundamental law of classical dynamics, Newton's second law, is independent of the observer, but, of course the first law, that of inertia, becomes false and, for that which concerns the apparent forces, so does the third law, that of action and reaction.

4. - Lagrangian relativity.

In Lagrange's mechanics the differential equations of the second order that refer to the motion of a mechanical system, subject to smooth bonds and to conservative active action, are obtained from a single function of the coordinates, of their derivatives with respect to time, and of time itself: « the Lagrangian function ». The equations thus obtained are invariant in those transformations in which, leaving the durations unaltered (because time is absolute and must maintain its privileged character), they allow a generic passage from some coordinates to others,

functions of the preceding ones and of time. Thus in particular in the case of a particle, these equations allow us to obtain the description of motion relative to an observer however mobile compared with an inertial observer.

Therefore Lagrange's mechanics satisfy a principle of general relativity of Cartesian type, because they concern only the differential laws of motion (and not all mechanical phenomena, like Galileo's more limited principle of relativity). The invariance of the differential equations is however subordinated to a « basic » function on which the change of the variables must be directly operated. The base is the « dynamic base » constituted by the function of Lagrange.

In the case of a particle this dynamic base involves (as in all the laws of classical physics) a geometrical base, made up by the square of the distance between two generic points infinitely near in space and, of course, a temporal base, made up of a generic infinitesimal time interval. One must however bear in mind that, whilst in orthogonal Cartesian coordinates, the geometric base is given by a tridimensional Pythagorean form, the generic transformation of coordinates changes it into a more general quadratic form. In spite of their generality, the considered transformations do not include those that involve the coordinates and time, those which, in the case of a particle, allow the passage from the 4 variables, made up by the 3 spacial coordinates and by time, to 4 other generic variables.

Similar to Lagrange's mechanics are Hamilton's mechanics, who substitutes Lagrange's differential equations of the second order with a double number of differential equations of the first order, the « canonical » equations, concerning the coordinates and the « kinetic moments ».

The invariance of these equations is due to the « canonical transformations », which are subordinated to the base constituted by Hamilton's function. This function represents the total energy, when the bonds are fixed, and it is interesting to note that it is dependent on time in the same way as the coordinates are on the corresponding kinetic moments. I shall come back to this point later on.

5. - Constitutive relativity.

In classical mechanics there are laws which satisfy Cartesius' principle of general relativity, without necessarily being subordinated to a base, as occurs for Lagrange's equations, or without need of introducing the rule of Coriolis, or that of the school of the thread, as occurs for the fundamental law of dynamics. Indeed the « constitutive laws » are indifferent to the observer: for instance, for elastic bodies, Hooke's law (« ut

tensio sic vis »), according to which the state of stress depends linearly and homogeneously on the state of deformation; for example, for perfect fluids, the law that considers stresses isotropic; for example, for viscous fluids, the law that considers the deviator of stress (which individuates their anisotropic part) as a homogeneous linear function of the tensor deformation velocity, etc. These laws do not vary passing from one observer to another, even if in accelerated motion relative to the former. When Galileo's great ship, instead of staying still, moves, even if the movement is not uniform but swaying this way and that, the laws on the behaviour of the materials, the constitutive laws, remain valid, remain the same, although their physical parameters vary.

6. - Einstein's special relativity.

Let us go back to restricted invariance in Galileo's sense, which makes it impossible to express with mechanical means the uniform rectilinear translatory movement of the observer in relation to another. This principle of relativity is valid not only for mechanical phenomena but also for electromagnetic phenomena, and therefore the propagation of light is the same for two observers in mutual uniform rectilinear translatory motion. This experimental fact is at the basis of relativity according to Einstein.

In the passage from one inertial observer to another also inertial, neither the distance dl between two infinitely near points, nor the elementary duration dt , remain unvaried, but the propagation of light does and with it the quadratic form:

$$(1) \quad ds^2 = c^2 dt^2 - dl^2 = dy_0^2 - (dy_1^2 + dy_2^2 + dy_3^2),$$

form which equaled to zero gives the law of propagation of light and its constant speed c , the same for both the inertial observers.

This leads to remarkable kinematic consequences: the contraction of the lengths and the dilatation of the times in the bodies in motion, innate in the transformations of Voigt-Lorentz which substitute the transformations of classical kinematics in the passage from one observer to another in translatory rectilinear uniform motion with respect to the former, if one wishes (1) and thus the propagation of light to remain invaried; the composition of the speeds according to a rule that leads to the impossibility of overcoming by composition the speed c of light.

Kinematics which are founded on the sole invariance of the propagation of light and limit the references to the inertial ones, for which the

pseudo-Pythagorean form subsists always and everywhere, are the « special relativistic kinematics ».

To find the corresponding dynamics, it is sufficient to use Lagrange's method, substituting the temporal element $dy_0 = c dt$, that does not have invariant character with the « proper time » element ds , given by (1), that possesses this characteristic. One must of course consider the Lagrange function dependent on 4 coordinates: the temporal coordinate $y_0 = ct$ and the three Cartesian spacial coordinates y_1, y_2, y_3 ; and also dependent on their four derivatives, not with respect to time, but to its proper time, dependent that is on the 4 components of the « relativistic velocity ». The equations thus obtained are 4: three spacial and one temporal one. The three spacial ones are analogous to those of Newton's classic dynamics. In them however the mass varies with velocity and tends to infinity when the velocity tends to that of light. The fourth equation can be considered a consequence of the other three and is the theorem of energy, as kinetic energy is given by the mass (variable) multiplied by c^2 , square of the light velocity. In this way mass and energy are different expressions of a same entity.

When the velocities are negligible compared with that of light, Einstein's mechanics coincide with Newton's classical mechanics, but when this does not occur it is well known how experiments confirm the consequences derived from Einstein's equations: especially the variability of the mass with the velocity and the equivalence between mass and energy.

7. - The space-time.

Einstein's theory of relativity denies that three-dimensional geometric space and time have separate absolute characteristics, independent that is of the observer, but, as a consequence of the invariance of the propagation of light, he attributes absolute character to the two together, to the « space-time » made up by the four-dimensional continuum the metrics of which, in relation to an inertial observer, are given by the pseudo-Pythagorean differential form (1) in the 4 variables y_0, y_1, y_2, y_3 .

Each elementary event, the combination of a spacial and a temporal determination, is represented by a point of space-time; each corpuscular movement by a space-temporal line.

As it is presumed that the law of inertia and that of restricted relativity are valid always and everywhere, its metrics can always and everywhere be expressed with the same pseudo-Pythagorean form, or, as one says, are constituted by a single « region of Galileo », and is therefore all pseudo-Euclidean. In this « space-time », introduced by Minkowski, space and time merge, but are not confused, because the temporal lines and the

purely spacial ones have metrics of opposite signs. Having, as in (1), the signs $+\dots-$, there are vectors which are time-like, whose norm is positive (like for example relativistic velocity) and others space-like whose norm is negative.

Like in ordinary spacial geometry, also in space-time one can refer, instead of to the inertial observer considered by (1), to a space-temporal reference which employs generic coordinates. It is sufficient to pass from the coordinates y which appear in (1) to general coordinates x . The metrics of course will lose the pseudo-Pythagorean form to take on the quadratic form:

$$(2) \quad ds^2 = g_{\alpha\beta}(x) dx^\alpha dx^\beta,$$

but will still remain, like (1), indefinite in sign.

The coordinates x , even if they cannot, like y , be given direct physical definition, nevertheless consent the determination of the concordance of events, and that is sufficient in physics.

It is thus possible, employing general coordinates, to operate in four-dimensional space-time, which constitutes the absolute of Einstein's physics, as in classical physics one operates in three-dimensional geometrical space, which constitutes its absolute. Physical states will thus be represented, in general coordinates, by scalars, vectors, space-time tensors, and physical laws by invariant, vectorial, tensorial laws, which remain valid whatever spacial-temporal reference is adopted.

In this way Einstein laid the basis of a physical theory satisfying the «principle of general relativity», making true Cartesius' dream of amplifying the geometric relativity.

In pseudo-Euclidean space-time the «law of inertia» is of geometrical nature, because the motion of an isolated particle is given by one if its straight lines, a geodetic. When the reference is general, the metrics take on the form (2) and from the geodetic law the inertial forces appear (the apparent drag forces and Coriolis'), forces which are represented by «Christoffel's symbols», built with the ordinary derivatives of the coefficients $g_{\alpha\beta}$ of metrics (2). These coefficients represent therefore the potentials of the forces, and make up a tensor, the «fundamental tensor»; but the forces themselves are represented by Christoffel's symbols, which are simple geometric objects, not making up a tensor, because they are null in the inertial reference considered by (1) (which indeed does not include apparent forces), but not in the general reference considered by (2).

The equations of «the dynamic field of a continuous body» have tensorial character in space-time, but not pertinent only to the latter's geometry (as for the law of inertia). They show that the flux which

diverges from the outer limit of each space-temporal region of a double symmetric tensor (marked by two indexes), the «energy tensor», is null. In the reference considered by (1), the component of this tensor with both the temporal indexes represents the density of energy, the three marked with one temporal index and the other spacial represent the density of momentum, the other six, marked with both the spacial indexes, the density of the kinetic energy and the stresses. The field equations are four: three, marked with a spatial index, express the theorem of momentum and the fourth, marked with a temporal index, the theorem of energy.

Electromagnetic laws also have tensorial character. The electromagnetic field is represented (in vacuum) by a single double antisymmetric tensor of space-time, which has six independent and not null components. In the reference presupposed by (1), the three components with one spacial and one temporal index represent the electrical field whilst the three with both spacial indexes represent the magnetical field. The electromagnetic tensor is irrotational, that is its work along any closed space-temporal surface is null, and its divergence equals the space-temporal vector which gives electrical distribution, that is the flux that diverges from the outer limit of each space-time region is equal to the electrical impulse contained in it.

The tensorial character in space-time that the theory of relativity has attributed to electromagnetic laws has rendered useless the introduction of a reference frame, which was the only remaining task left to ether.

On the subject of the dynamics of continuous bodies and of electrodynamics it must be noted that their relativistic laws are to be obtained from those on which their statics are founded, substituting three dimensional geometric space with the four dimensional space-time. We may discern in this suggestive circumstance the relativistic aspect of d'Alembert's principle, which allows the passage from statics to dynamics in classical mechanics.

8. - Einstein's theory of gravitation.

As gravitational mass coincides with inert mass, there is complete equivalence between uniform gravitation relative to an inertial observer and inertia relative to an observer (not inertial) in opportune uniformly accelerated motion relative to the former. For this reason a theory of general relativity, like that of Einstein, precisely as consequence of the principle on which it is founded, must also be a theory of gravitation. In it indeed inertia and gravitation must blend into a single concept.

In any restricted (infinitesimal) space-time region an inertial reference is to be found where gravitation is absent and the laws of special relativity are valid. Such a reference is, for example, that of an observer inside the cabin of a satellite, when, once the action of the rocket ceases, it is subject only to the attraction of the earth: for it, weight does not exist.

The conditions imposed make it so that the space-time is made up of a series of pseudo-Euclidean elements, but so connected that the space-time, in its unity, ceases, in the presence of a generic gravitational field, to be pseudo-Euclidean. We can however consider it « Riemannian », such, that is, that the connection between its elements is completely analogous to that which takes place in an ordinary curved surface, apart, of course, from the circumstance that the elements of an ordinary surface have 2 dimensions, whilst those of the space-time have 4.

The geometry of a Riemann space is individuated entirely by its metrics. These however cannot be put everywhere in the Pythagorean (or pseudo-Pythagorean) form, if the space is not Euclidean (or pseudo-Euclidean). The tensor field $g_{\alpha\beta}(x)$ which individuates the metrics of the space-time must therefore depend on the gravitational field and, through it the gravitational phenomena will influence the space-time which contains them.

It is to be noted that in a Riemann space-time also the tridimensional geometric space included in it results to be Riemannian, but generically non-Euclidean, as on the other hand the geometric space in a pseudo-Euclidean space-time may result non-Euclidean, when the reference is not inertial, and so the metrics do not have pseudo-Pythagorean form.

In Einstein's theory gravitation has a geometric interpretation identical to inertia. A geodetic of the space-time gives the gravitational motion of a « test particle », conformly to the circumstance that gravitational motion, like inertial one, is independent of the mass of the mobile particle (at least if this mass is small enough not to influence the motion of masses to which the field is due). The fundamental tensor $g_{\alpha\beta}$ represents the gravitational and inertial potentials, and Christoffel's symbols, constructed with their ordinary derivatives, individuate the gravitational and inertial forces.

The various forms of gravitational energy influence the geometry of the space-time through a double symmetrical tensor $R_{\alpha\beta}$. It is obtained by contracting Riemann's tensor, which represents the curvature and is constructed with Christoffel's symbols and their derivatives.

Considering that, externally to the gravitating masses, the space-time is indeed curved, but in such a way that the mean value of the local mean curvature results stationary, which is to be obtained by

contracting the tensor $R_{\alpha\beta}$, one obtains the «gravitational equations»: 10 second order partial differential equations, in the 10 components of the tensor $g_{\alpha\beta}$. Externally to the masses, they simply show that

$$(3) \quad R_{\alpha\beta} = 0.$$

Internally to the masses the gravitational equations show that the energy-momentum tensor $T_{\alpha\beta}$ is proportional (having a universal constant χ as coefficient of equivalence) to a double symmetrical tensor $A_{\alpha\beta}$, linearly and homogeneously dependent on $R_{\alpha\beta}$:

$$(4) \quad A_{\alpha\beta} = \chi T_{\alpha\beta}.$$

The field equations, (3) external and (4) internal, are in agreement with astronomic observation (displacement of the perihelion of the planets, displacement of luminous rays and displacement of the spectral lines towards red as effect of a gravitational field); they allow us to successfully face cosmological problems having spacially finite universe models, but unlimited and in expansion; they give a justification of Mach's principle, according to which inertia, like gravitation, is due to a mutual action between bodies.

9. - Consequences of the principle of general relativity.

An important consequence of the principle of general relativity is the following: if in the variational principle that gives origin to the field equations (3) a variation is carried out which corresponds to a generic change of coordinates, the principle must nevertheless be identically satisfied. With this procedure of Weyl the 4 identities of Bianchi are obtained, by which the divergence of the tensor $A_{\alpha\beta}$ which appears in (4) is null. From this identity there follows, owing to (4), the vectorial equation which shows that the divergence of the energy tensor $T_{\alpha\beta}$ is null, and this equation, that expresses a theorem of conservation, gives the laws of the dynamic field of matter.

In particular, when the energy tensor is the one that corresponds to disgregated matter, from the field equations (4) there derives the law of the space-time geodesic. In every case therefore the laws of the motion of matter are consequences of the field equations, owing to the principle of general relativity.

Another consequence of this, as Fock proved, is that the motion of a bunch of matter does not depend on its inner structure, and this is true even if the bunch is a punctiform singularity of the field. Directly from (3), bearing in mind Bianchi's identities, derived from the principle

of relativity, Einstein, Infeld and Hoffmann worked out the differential equations that give the motion of singularities of the field itself. In particular, also in this way, the motion of a test particle results as being given by a geodetic of the space-time curved by the other masses, and the law of the space-temporal geodetic (obvious extension of the law of inertia) is a consequence of the equations of the gravitational field. From the mathematical point of view it is interesting to note the circumstance that from the partial differential equations (3), making up the field equations, the ordinary differential equations, that give the motion of the singularities which are the sources of the field, can be derived. From the relativistic point of view however one must observe that the laws of motion thus found lose the property of invariance which the geometric or tensorial laws in space-time have, because the dependence of the three spacial coordinates on the time coordinate leads to a choice of the system of coordinates.

There is another important consequence of the principle of relativity, when the laws that govern a physical field are formulated stating that an «action» is stationary.

In the classical extremum principle that governs a mechanical field, the action is given by the integral relative to the time of the spacial triple integral of that energetic density that makes up Lagrangian density. Similarly in the variational principle, which generalizes Hamilton's principle, adapting it to Einstein's theory, the action becomes the integral (still quadruple) extended to a space-time region of an energetic density (representing Lagrangian space-time density). Such density is constructed with those space-time tensors that characterize the physical field. As in so doing the action remains invariant when we carry out an infinitesimal change of the space-time coordinates, in correspondence with such transformations laws of conservation are obtained, owing to the general theorem of Emmy Noether. Indeed this theorem states that when in a physical theory the laws are derived from a variational principle that includes an invariance group, to each invariant transformation there corresponds a quantity that is conserved.

Let us consider therefore an infinitesimal displacement ξ_α in space-time. To it corresponds an infinitesimal transformation of coordinates and the consequent variation $\delta g_{\alpha\beta}$ of the fundamental tensor is the double of the symmetrical part of the tensor derived from the displacement:

$$(5) \quad \delta g_{\alpha\beta} = \xi_{\alpha|\beta} + \xi_{\beta|\alpha}.$$

Correspondingly, from the variational principle which governs the physical relativistic field, we can derive a theorem of conservation, expressed by the annullment of the divergence of a space-time tensor.

10. - Unitary relativistic theories.

Einstein's theory of general relativity gives a geometric model of inertia and gravitation, model belonging to the Riemannian space-time which it employs. In it the electromagnetic field indeed finds adequate tensorial representation and the laws which govern it are tensorial, but the field and the laws find no geometrical representation. From this the wish has come to have a «unitary» relativistic theory which, employing a continuum having a ampler geometry than that of the Riemannian space-time, could give a geometrical model both of gravitation and of the electromagnetic field, and in substance therefore of all the phenomena of the macrocosm. Kaluza's theory tried to make this wish come true employing a five-dimensional Riemannian continuum, and Einstein's last unitarian theory tried too, employing a four-dimensional non Riemannian space-time, with a nonsymmetrical fundamental tensor. To geometrize physics has indeed always been the dream of the philosophers of nature, from Plato who said *ΑΕΙ Ο ΘΕΟΣ ΓΕΟΜΕΤΡΕΙ* (God always geometrizes) to Jacobi, Lipschitz, Hertz, Darboux, Ricci, Levi-Civita, who tried to geometrize dynamics (not however in space-time).

Also in unitary relativistic theories important consequences are derived from the principle of relativity, as in the only gravitational theory. Thus in Einstein's last unitary theory from the variational principle that (like in the gravitational theory) expresses the stationary quality of the mean value of the mean local curvature, giving the field equations we derive identities analogous to those of Bianchi. A consequence is the determination of the motion of the sources of the field.

11. - The problem of energy.

Physical laws are entirely consonant to the principle of general relativity, both when they are concerned with the geometry of the space-time, and when they are concerned with space-time fields. And if we consider that also the geometry of the space-time is individuated by the field $g_{\alpha\beta}(x)$ of its fundamental tensor, we can indeed say that the field concept (which Einstein considered man's greatest conquest in science) dominates the theory of relativity.

The relativistic fields have in common a fundamental quality: that of merging in a single tensorial entity, quantities that classical physics keep separate. In the fundamental tensor field $g_{\alpha\beta}(x)$ the spacial determinations merge with the temporal one, the inertial potentials with the gravitational ones; in the field $T_{\alpha\beta}(x)$ of the energy-momentum tensor

there merge density of energy, due to the distribution of mass and to its (kinetic) motion, density of momentum and stresses; in the electromagnetic field, in the field that is of the only antisymmetric tensor of space-time that individuates it, the magnetic and the electric fields merge; in the field of the space-time vector which gives electric distribution, electric density and the specific electric current merge. The equations of the relativistic fields synthesize different classical laws: the law of inertial motion and that of gravitational motion are expressed by the same law of the space-temporal geodesic; the law of momentum and that of energy merge in one only law which expresses the annihilation of the divergence of the energy tensor; the first law of Maxwell and the one that denies the existence of free magnetism merge in the irrotational character of the electromagnetic tensor; the second law of Maxwell and the one of the conservation of electricity merge in the law that states that the divergence of the electromagnetic tensor is equal to the electric distribution.

However if always and everywhere we use a same inertial reference (a thing which is possible only if the space-time is pseudo-Euclidean), the time can be distinguished from the spacial coordinates, thanks to the different signs of the one and of the others in (1). In this case one can separate also the density of energy from that of the momentum and from the stresses, the electric field from the magnetic one, the density of electric charge from the specific current, the law of the momentum from that of energy, not to mention the various electromagnetic laws.

But this separation is not so immediate when the space-time is not pseudo-Euclidean and one cannot always and everywhere use the same inertial reference: energy and momentum, as well as the theorems relative to these, then correspond simply to different components of same tensor, which depend on the reference, and the same thing occurs for the electric field and the magnetic one, for electric density and specific current, not to mention the electromagnetic laws that refer to them.

That of separating energy from the momentum and separating the corresponding laws is a matter from which the physicist cannot become detached little as they may wish to use intuition, the heritage of three centuries: the XVII and the XVIII dominated by the law of the momentum, the XIX by the conservation of energy.

The concept of energy in the theory of relativity does not have universal value, but there are certainly cases in which it takes on full meaning. If in the space-time there exists a field of Killing (that is a vectorial field $\xi_\alpha(x)$ the 4 components of which satisfy the 10 equations summed up in (5) in which $\delta g_{\alpha\beta} = 0$) and if the vector ξ_α is time-like (that is has positive norm), then the corresponding transformations allow us

to determine systems of coordinates by which we can indeed speak of energy and of its conservation (and thus also, complementarily, of a theorem of momentum). More generally it is possible to define a density of energy if the physical situation is connected to vectorial fields of space-time intrinsically defined like time. This intimate interdependence between time and energy finds its roots in Hamiltonian mechanics as I have said.

The problem of energy in the theory of relativity is still open and it is the first theme of the Meeting on Relativity. From the very general point of view of Emmy Noether's theorem it can be expressed as follows: from the infinite infinitesimal transformations of space-time coordinates, characterized through (5) by a displacement ξ_α , to each of which corresponds a conservation theorem, extract those that imply the conservation of a quantity that can be interpreted as energy.

12. - The problem of gravitational radiation.

The second problem to which the Meeting on Relativity is dedicated is that of gravitational waves.

The equations (3) or (4) of the gravitational field allow some real characteristic varieties, through which there may be discontinuities in the second derivatives of the potentials. These fronts propagate with the velocity of light, and therefore it is with this velocity that the gravitation actions propagate. It is important to note that precisely because there is an effective propagation of light, the space-temporal metrics are indefinite in sign, making it possible to distinguish space from time, and it is exactly thanks to this circumstance that the characteristic varieties are real and the gravitational actions can propagate.

This identity between luminous wave fronts and gravitational wave fronts leads naturally to the problem of examining if the equations of the gravitational field allow, like the equations of the electromagnetic field, which govern the propagation of light, solutions corresponding to a wave propagation, making up a gravitational radiation.

The problem depends essentially on the data attributed to the hypersurface of the space-time which corresponds to the initial instant and to the asymptotic conditions.

* * *

In announcing the two themes of the Meeting on Relativity my task ends. It was limited to a description, simple and conversational if possible, of the extension and the evolution of the principle of relativity

from Galileo to Einstein, and of the important consequences that have derived from it.

Kindly forgive me for all that I have not said, if what I have said has sometimes been vague, sometimes involved, and forgive me if no less than five times I have used mathematical formulas; I can but turn for justification to Galileo who in his *Saggiatore* said that the book of nature «è scritto in lingua matematica, e i caratteri sono triangoli, cerchi ed altre figure geometriche, senza i quali mezzi è impossibile intenderne umanamente parola». And among these « words » please include my five formulas.

Energy and Momentum Carried by Gravitational Waves. (*)

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In general relativity matter is inseparably connected with gravitational fields through Einstein's field equations. For this reason one cannot expect to have strict conservation laws of energy and momentum for the matter system alone. But, at an early stage it was shown by Einstein, Lorentz, F. Klein and others [1] that conservation laws exist for the complete system of matter plus gravitational field of the form

$$(1) \quad T_{i,k} \equiv \frac{\partial T_i^k}{\partial x^k} = 0 .$$

Here

$$(2) \quad T_i^k = \mathfrak{T}_i^k + t_i^k$$

is the sum of the energy-momentum tensor density $\mathfrak{T}_i^k$ of the matter and a quantity t_i^k which depends on the gravitational field variables only. It is apparent from the form of (1) that $\mathfrak{T}_i^k$ or rather the gravitational part t_i^k cannot be a true tensor density. In fact it is only an *affine* tensor density for which Lorentz coined the name *energy momentum complex*.

Einstein was the first to give an explicit expression for the complex t_i^k in terms of the components of the metric tensor and its derivatives, and he could show that this expression leads to satisfactory formulae for the total energy and momentum of a closed system in a certain class of co-ordinate systems. As regards the distribution of the energy throughout space he came to the conclusion that this notion has no exact physical meaning in general relativity, and this also seems to be in accordance with the actual situation regarding the measurability of the gravitational energy inside a small region of a physical system.

(*) This paper was written during the author's stay at the University of Texas, Austin, in the spring term 1964. I want to express my sincere thanks to Dr. A. Schild and his collaborators for the most pleasant and inspiring time spent at the Relativity Centre in Austin.

However, even if one confines oneself to the consideration of the total energy of a system and its possible variation in time, such as in calculations of the energy emission from an insular system, Einstein's expression is not quite satisfactory. First of all it is not in accordance with the general principle of relativity according to which any relation between measurable physical quantities, like the total energy and the components of the metric tensor, must have the same form in all systems of space-time co-ordinates. As first pointed out by Bauer [2] this is most drastically demonstrated when one considers the trivial case of a completely empty world with a flat space-time, where Einstein's expression gives an infinite value for the total energy when calculated in polar co-ordinates, in contrast to the correct value zero obtained if one uses Cartesian co-ordinates. We get similar meaningless results in most other systems of co-ordinates. Therefore it becomes in general a problem to decide in which systems of co-ordinates Einstein's expression is applicable and for this one actually needs a generally covariant expression with which to compare.

In two recent papers [3, 4] in the Communications of the Danish Academy a new expression for the energy-momentum complex T_i^k was derived which gives the total momentum and energy for an arbitrary insular system in complete accordance with the general principle of relativity. In contrast to Einstein's old expression which was a function of the metric tensor and its derivations the new expression for T_i^k (or t_i^k) is given in terms of tetrad field vectors $h^{(a)}_i$ connected with the metric by the relations

$$(3) \quad h^{(a)}_i h_{(a)k} = g_{ik} .$$

(The index a which runs from 1 to 4 labels the four tetrad vectors in each space-time point, and it is raised and lowered according to the special relativity rule for raising and lowering indices).

If we introduce the tensors

$$(4) \quad \begin{cases} \gamma_{ikl} = h^{(a)}_i h_{(a)k;l} = -\gamma_{kll} \\ \Phi_k = \gamma^i_{ki} = h^{(a)i} h_{(a)k;i} \end{cases}$$

and the symbols

$$(5) \quad \Delta^i_{kl} = h^{(a)i} h_{(a)k;l} , \quad |h| = \sqrt{-g}$$

the next complex t_i^k can be written

$$(6) \quad t_i^k = \frac{|h|}{\kappa} [\gamma^{km}_l \Delta^l_{mi} - \Phi^l \Delta^k_{il} + \Delta^l_{li} \Phi^k] - \frac{1}{2\kappa} \delta_i^k \mathfrak{L}$$

where

$$(7) \quad \mathfrak{L} = |h| [\gamma_{rst} \gamma^{tsr} - \Phi_r \Phi^r]$$

is the new Lagrangian of the gravitational field. Further, if $\mathfrak{L}_i^k$ in (2) is eliminated by means of Einstein's field equations

$$(8) \quad \mathfrak{G}_i^k = \mathfrak{R}_i^k - \frac{1}{2} \delta_i^k \mathfrak{R} = -\kappa \mathfrak{L}_i^k$$

the complex T_i^k can be shown to take the form

$$(9) \quad T_i^k = u_i^{kl}{}_{,l}$$

with the superpotential

$$(10) \quad u_i^{kl} = -u_i^{lk} = \frac{|\hbar|}{\kappa} [\gamma_i^{kl} - \delta_i^k \phi^l + \delta_i^l \phi^k].$$

It is an essential feature of the present formulation that u_i^{kl} is a true tensor density in contrast to the superpotential of von Freud [5], corresponding to Einstein's expression, which is an affine tensor density only.

Besides the formulae (2)-(10) we need a further assumption regarding the asymptotic behaviour of the tetrad field at spatial infinity. The form of these boundary conditions will depend both on the type of matter system considered and on the system of co-ordinates used. Let us consider an insular system where the matter is confined to a finite part of space (a timelike tube with a finite spatial cross section in 4-space). For such systems we may assume that space-time is asymptotically flat at large spatial distances from the system. In that case it is natural to require that the tetrads $h^{(a)}_i$ asymptotically tend to the values which hold for a completely empty space where the curvature tensor is zero everywhere.

It is now essential to avoid the difficulties mentioned above which met the Einstein expression in the case of a completely empty space. Therefore, we have to require that in this case T_i^k is zero everywhere and in every system of co-ordinates which is possible only if

$$(11) \quad \gamma_{ikl} = 0$$

everywhere. Thus, for a completely empty space we have to assume that besides the curvature also the «torsion» of the space (or the tetrad lattice) is zero. If we use rectilinear co-ordinates in a completely empty world, (11) implies that the components of the tetrad vectors must be constant. In particular in a Lorentz system of co-ordinates where

$$(12) \quad g_{ik} = \eta_{ik}$$

we can take

$$(13) \quad h^{(a)}_i = \delta^a_i$$

In the case of an insular system we can use an asymptotically Lorentzian system where g_{ik} tends to η_{ik} with increasing spatial distance. Then we shall require that the tetrads $h^{(a)}_i$ asymptotically tend to the constant values (13). Further, as regards the way in which $h^{(a)}_i - \delta^a_i$ go to zero with increasing spatial distance we impose the natural boundary condition that these quantities asymptotically behave in the same way as the metric quantities $g_{ik} - \eta_{ik}$. Otherwise the tetrad field can be chosen freely without, of course, violating the connection (3) with the metric. For a given metric field g_{ik} the tetrad field is by far not uniquely determined by (3), because independent Lorentz rotations of the tetrads in the different space-time points leave the left hand side of (3) unchanged. Thus, the tetrads are only determined by the metric up to independent Lorentz rotations

$$(14) \quad \check{h}^{(a)}_i = \Omega^{(a)}_{(b)}(x) h^{(b)}_i$$

where the coefficients $\Omega^{(a)}_{(b)}$ are arbitrary scalar functions satisfying the orthogonality relations at each point (x) .

The complex $T_i{}^k$ is in general not invariant under the group of rotations (14) except if the coefficients $\Omega^{(a)}_{(b)}$ are constants. This may be taken as an indication that the « energy-density » — $T_4{}^4$ is not a measurable physical quantity. On the other hand it was shown in ref. [4] that the total momentum and energy contained in a sufficiently large volume V , *i.e.*

$$(15) \quad P_i = \int_V T_i{}^4 dx^1 dx^2 dx^3$$

is invariant under all transformations (14) which respect the boundary conditions formulated above. The same holds true for the asymptotic form of $T_i{}^k$ at sufficiently large distances which determines the radiated energy and momentum. From this point of view the tetrads are subsidiary quantities which play a similar role as the potentials in electrodynamics and the transformations (14) have the character of gauge transformations which leave the measurable physical quantities unchanged.

The description of energy and momentum in this formulation is some what analogous to the case of the generally covariant-Dirac equation where it is necessary (or at least convenient) to describe the influence of gravitational fields on a fermion field by means of tetrads. Also in this case the measurable physical quantities like the charge-and current

densities are invariant under the transformations (14) while the wave functions themselves are not invariant. The only difference is that in our case the «wallowed» transformations (14) are somewhat restricted at spatial infinity by the boundary conditions for the tetrads formulated on p. 32.

Now let F' be the surface of the volume V in (15) and let $\Phi(F', t)$ denote the 2-surface in 4-space containing the events in F' at time t in a definite system of co-ordinates. Then, by applying (9) and Stoke's theorem in 4-space the quantities P in (15) may be written

$$(16) \quad P_i = -\frac{1}{2} \int_{\Phi(F', t)} u_i^{kl} dS_{kl} = -\frac{1}{2} \int_{\Phi(x, t)} A_i(x)$$

with

$$(17) \quad dS_{kl} = \delta_{iklm} dx^i \delta x^m$$

$dx^i, \delta x^m$ infinitesimal vectors lying in the surface $\Phi(F', t)$. In a third paper in *Nucl. Phys.*, ref. [6], it was shown that *the four momentum for a non-radiative system is a true free 4-vector* provided F' is chosen so large that $\Phi(F', t)$ lies entirely in a region where space-time may be considered as flat. This important property rests on the fact that u_i^{kl} is a true tensor density so that

$$(18) \quad A_i(x) = u_i^{kl} dS_{kl}$$

is a true vector at the point (x) on Φ and, further, that the sum of the vectors $A_i(x)$ can be given a unique geometrical meaning by parallel displacement of the vectors $A_i(x)$ to a common point p in the flat region. In a system of co-ordinates which is asymptotically Lorentzian the components of the four-momentum $P(p)$ are independent of p and simply equal to the algebraic sums of the components $A_i(x)$, *i.e.* the integrals (16). In a restricted class of asymptotically Lorentzian systems *viz.* those for which g_{ik} tends sufficiently rapidly to η_{ik} with increasing spatial distance, Einstein's expression P_i^E for the four-momentum gives the same result as (16). If the system of co-ordinates is not asymptotically rectilinear (for instance in a rotating system of reference) Einstein's expression becomes meaningless while the components of $P(p)$ in our formulation are obtained simply by means of the transformation law of a covariant vector at the point p .

In ref. [4] the gravitational energy and momentum radiated from an axi-symmetric matter system with out-going radiation only was calculated by means of the asymptotic solution of Einstein's field equations given by Bondi *et al.*, ref. [7]. In the present paper we shall sketch the

corresponding calculation for an arbitrary matter system by means of the asymptotic solution obtained by Sachs, ref. [8], for a radiating system without any symmetries. In a certain class of asymptotically Lorentzian co-ordinates

$$(19) \quad x^i = \{x, y, z, t\}, \quad r = \sqrt{x^2 + y^2 + z^2}$$

the metric tensor g_{ik} has at large spatial distances the form of a series expansion in $1/r$. If θ and φ are the « polar angles » and

$$(20) \quad \begin{cases} n_i \equiv \frac{x^i}{r} = \{\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta\} \\ u = t - r \end{cases}$$

the asymptotic solution of Sachs may be written

$$(21) \quad g_{ik} = \eta_{ik} + y_{ik} + z_{ik} + O_3$$

with

$$(22) \quad y_{ik} = \frac{\alpha_{ik}}{r}, \quad z_{ik} = \frac{\beta_{ik}}{r^2}$$

$\alpha_{ik}(u, \theta, \varphi)$, $\beta_{ik}(u, \theta, \varphi)$ functions of u , θ and φ only. Further, generally O_n means a term which goes to zero as r^{-n} for $r \rightarrow \infty$. If we introduce the quantities

$$(23) \quad \begin{cases} n_i = \{n_i, 0\} \\ m_i = \{\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta, 0\} \\ l_i = \{-\sin \varphi, \cos \varphi, 0, 0\} \\ \mu_i = \eta_i^4 - n_i, \quad t_i = m_i - il_i \end{cases}$$

the explicit expression for α_{ik} and β_{ik} are

$$(24) \quad \alpha_{ik} = 2\Re\{ct_it_k\} + \Re\{\psi(t_i\mu_k + \mu_it_k)\} + 2M\mu_i\mu_k$$

$$(25) \quad \begin{aligned} \beta_{ik} = & 2|c|^2(m_im_k + l_il_k) + \frac{1}{2}|c|^2(n_i\mu_k + \mu_in_k) - \\ & - \Re\{(2N + \frac{1}{2}\Delta|c|^2)(t_i\mu_k + \mu_it_k)\} + \\ & + \Re\{\Delta^*(N \sin \theta)\}\mu_i\mu_k. \end{aligned}$$

Here Δ is the operator

$$(26) \quad \Delta = \frac{\partial}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial}{\partial \varphi}$$

and

$$(27) \quad \psi = \frac{1}{\sin^2 \theta} \Delta^*(c \sin \theta) .$$

Further, $c(u, \theta, \varphi)$, $N(u, \theta, \varphi)$ are complex functions and $M(u, \theta, \varphi)$ is a real function of u, θ and φ which depend on the type of matter system considered. They are connected by the « supplementary » relations

$$(28) \quad \begin{cases} M_0 = -|c_0|^2 + \frac{1}{2} A_0 \\ 3N_0 = -\Delta M - (4c \cos \theta + \Delta c + 3c\Delta)c_0^* \end{cases}$$

where

$$(29) \quad A = \frac{1}{\sin \theta} \Re \{ \Delta^*(N \sin \theta) \}$$

and the index zero indicates partial derivation with respect to the retarded time u . Further, the star means complex conjugation and $\Re \{ \}$ denotes the real part of a complex function. In contrast to the axi-symmetric case the « news function » c_0 is here complex, *i.e.* there are actually two independent real news functions in the general case considered by Sachs.

Like in the axi-symmetric case treated in ref. [4] we can now for large r choose the tetrads

$$(30) \quad h_{(a)i} = \eta_{ai} + \frac{1}{2} y_{ai} + \frac{1}{2} (z_{ai} - \frac{1}{4} y_{ar} y^r_i) + O_3$$

and the calculation of the complex T_i^k and the superpotential u_i^{kl} for large r runs exactly as in ref. [4]. For T_i^k we get (comp. eq. (4.37) in ref. [4])

$$(31) \quad \begin{cases} T_i^k = t_i^k = \frac{1}{4\kappa r^2} (\alpha_{lm})_0 (\alpha^{lm})_0 \mu_i \mu^k + O_3 = \\ = \frac{2|c_0|^2}{\kappa r^2} \mu_i \mu^k + O_3 . \end{cases}$$

The difference between (31) and the corresponding formula in the axi-symmetric case is only that the square of the real news function is replaced by the absolute square of the complex quantity c_0 . For sufficiently large values of r the last term in (31) may be neglected and we get for the amount of gravitational energy emitted through a surface element $(d\theta, d\varphi)$ of a large « sphere of radius r »

$$(32) \quad S d\theta d\varphi = -T_4^k n_k r^2 \sin \theta d\theta d\varphi = \frac{2|c_0|^2}{\kappa} d\Omega$$

on the analogy of (4.41) in ref. [4]. The total energy which per unit time leaves the sphere of radius r is, therefore,

$$(33) \quad -\frac{dH}{dt} = \int S \, d\theta \, d\varphi = \frac{1}{2} \int_0^\pi |c_0|^2 \sin \theta \, d\theta .$$

For the four-momentum contained in a large sphere of radius r at time t one gets from (16)

$$(34) \quad P_i = \int u_i^{4j} n_j r^2 \, d\Omega = - \int u_i^{4l} \mu_l r^2 \, d\Omega .$$

By a calculation exactly analogous to the calculation performed in the Appendix of ref. [4] one finds for the integrant in (34)

$$(35) \quad -u_i^{4l} \mu_l = \frac{1}{2\kappa r^2} [(-4M + \Re\{A\})\mu_i - \Re\{\Psi t_i\}] + O_3 .$$

Again we may neglect the term O_3 for sufficiently large values of r and, by partial integration, it is easily seen that

$$(36) \quad \int A \mu_i \, d\Omega = \int \Psi t_i \, d\Omega = 0 .$$

Hence

$$(37) \quad P_i = -\frac{2}{\kappa} \int M \mu_i \, d\Omega = \{P_i, -H\} .$$

Thus, the «total» momentum and energy is determined by the function $M(u, \theta, \varphi)$ through

$$(38) \quad \begin{cases} P_i = \frac{2}{\kappa} \int M(u, \theta, \varphi) n_i \, d\Omega \\ H = m(u) = \frac{1}{2} \int_0^\pi M(u, \theta, \varphi) \, d\Omega . \end{cases}$$

The latter equation is in accordance with *definitions* of Bondi and Sachs.

The proof of the invariance of the total four-momentum as well as of the asymptotic expression for T_i^k under all «gauge transformations» (14) which respect the boundary conditions for the tetrad field runs exactly as in the axi-symmetric case (see Sect. 6 of ref. [4]).

The time derivative of P_i is, by (38), (28), (36) and (32)

$$(39) \quad \frac{dP_i}{dt} = \frac{2}{\kappa} \int M_0 n_i d\Omega = -\frac{2}{\kappa} \int |c_0|^2 n_i d\Omega = -\int S n_i d\theta d\varphi$$

which shows that the emitted gravitational radiation administers a recoil to the system of the same amount as if it were electromagnetic radiation.

A non-radiative system is characterized by the vanishing of the news function c_0 . This does not necessarily mean that the system is stationary, but certainly M_0 must vanish on account of (28). Thus, in this case P_i in (37) is independent both of t and of the radius r provided r is sufficiently large. Actually P_i is then also independent of the form of the surface F .

In an asymptotically Lorentzian system of co-ordinates of the type (19), where g_{ik} is of the form (21)-(25) Einstein's energy-momentum complex gives the same value for the four-momentum as the complex (9), i.e. we have

$$(40) \quad P_i^s = P_i$$

in such systems. In more general systems of co-ordinates this is not the case. We know already that (40) does not hold in systems which are not asymptotically rectilinear where P_i^s becomes meaningless. However, if an asymptotically Lorentzian system is defined as *any* system in which $g_{ik} \rightarrow \eta_{ik}$ with increasing spatial distance the eq. (40) is not even true in all such systems of co-ordinates. Let us, starting from the co-ordinates (19), introduce new co-ordinates $\bar{x}^i$ by

$$(41) \quad \bar{x}^i = x^i + \xi^i(x) .$$

In order that the new system be again asymptotically Lorentzian we shall assume that asymptotically

$$(42) \quad \xi_{,k}^i \rightarrow 0 \quad \text{for } r \rightarrow \infty .$$

If we further assume that

$$(43) \quad (\xi^i/r) \rightarrow 0 \quad \text{for } r \rightarrow \infty .$$

the new radial co-ordinate $\bar{r}$ defined by

$$(44) \quad \bar{r}^2 = \bar{x}^i \bar{x}_i = r^2 \left(1 - \frac{2n_i \xi^i}{r} + \frac{\xi^i \xi_i}{r^2} \right)$$

will also approach r at spatial infinity:

$$(45) \quad \bar{r} \rightarrow r \quad \text{for } r \rightarrow \infty .$$

Similarly we have

$$(46) \quad \bar{n}_i = \frac{\bar{x}^i}{\bar{r}} \rightarrow n_i \quad \text{for } r \rightarrow \infty .$$

In calculating the four-momentum in the new system

$$(47) \quad \bar{P}_i = - \int \mathfrak{U}_i{}^{4\lambda} \bar{n}_\lambda \bar{r}^2 d\bar{\Omega}$$

we only need the values of the superpotential at large spatial distances where the transformation (41) can be treated as an infinitesimal transformation on account of (42). Since $\mathfrak{U}_i{}^{kl}$ is a true tensor density and $\mathfrak{U}_i{}^{kl} = O_2$ we get then

$$(48) \quad \bar{\mathfrak{U}}_i{}^{kl} = \mathfrak{U}_i{}^{kl} + O_2$$

where σ_n is a quantity which goes to zero *faster* than r^{-2} for $r \rightarrow \infty$. Therefore, on account of (45), (46) the difference $\bar{P}_i - P_i$ goes to zero with increasing radius and we conclude that the total four-momentum of a nonradiative system is invariant under all transformations of the type (41)-(43):

$$(49) \quad \bar{P}_i = P_i .$$

Now, it is easily seen that the Einstein quantities $P_i{}^{\mathcal{E}}$ in general do not have this property because the superpotential of von Freud is not a true tensor density. Take for instance the case where

$$(50) \quad \begin{cases} \xi^i(x) = x^i f(r) & \zeta^4 = 0 , \\ f(r) \rightarrow 0 & \text{for } r \rightarrow \infty . \end{cases}$$

Then, a straight forward calculation of the metric and the superpotential of von Freud at large spatial distances gives for the total energy in the new system of co-ordinates according to Einstein's expression

$$(51) \quad \bar{H}^{\mathcal{E}} = - \bar{P}_4{}^{\mathcal{E}} = H^{\mathcal{E}} + \frac{1}{2} r^3 (f'(r))^2 |_{r \rightarrow \infty} .$$

For $r \rightarrow \infty$ the last term may take on any value between zero and infinity depending on how $f(r)$ goes to zero with increasing r . Since this term does not depend on the parameters of the matter system it would appear also in a completely empty space which shows that we are concerned with a new form of the old Bauer difficulty. The term in question becomes zero only if $f(r)$ tends to zero faster than $r^{-\frac{1}{2}}$ for $r \rightarrow \infty$. Thus, we arrive

at the conclusion that Einstein's energy-momentum complex gives correct values for the total four-momentum in those asymptotically Lorentzian systems of co-ordinates only for which $g_{ik} - \eta_{ik}$ tends to zero faster than r^{-1} . In all other systems of co-ordinates this complex leads to meaningless results.

In contrast to the case of a non-relative system, where the four-momentum P is a free 4-vector, the transformation of P for a radiative system is in general quite complicated even under linear space-time transformations, ref. [4]. This is not surprising, for already in special relativity the 4-vector character of the four-momentum is closely connected with the conservation of this quantity.

We shall now investigate the character of the gravitational waves emitted by an arbitrary matter system. Consider a spatial region V of linear dimensions l at a very large distance R from the matter system so that

$$(52) \quad l \ll R.$$

If V is centered around a point $x = R$, $y = z = 0$ on the positive x -axis of the system of co-ordinates (19) it is convenient to introduce co-ordinates

$$(53) \quad \bar{x}^i = \{\bar{x}, \bar{y}, \bar{z}, \bar{t}\} = \{x - R, y, z, t\}.$$

Exactly as in Sect. 8 of ref. [4] it is seen that the metric (21)-(25) inside V takes the form of a weak field expansion

$$(54) \quad \bar{g}_{ik} = \eta_{ik} + \overset{(1)}{g}_{ik} + \overset{(2)}{g}_{ik} + \dots$$

where the first term has the character of a plane wave, *i.e.* y_{ik} is a function of

$$(55) \quad \bar{u} \equiv \bar{t} - \bar{x}$$

only. In fact we get from (22)-(24) inside V for sufficiently large values of R

$$(56) \quad \overset{(1)}{g}_{ik} = 2\Re\{\bar{c}(\bar{u})\bar{t}_i\bar{t}_k\} + \Re\{\bar{\psi}(\bar{u})(\bar{t}_i\bar{\mu}_k + \bar{\mu}_i\bar{t}_k)\} + 2\bar{M}(\bar{u})\bar{\mu}_i\bar{\mu}_k$$

where $\bar{M}(\bar{u})$, $\bar{c}(\bar{u})$ and $\bar{\psi}(\bar{u})$ are the functions of $\bar{u}$ obtained from M/r , c/r , ψ/r in this limit. Further, $\bar{\mu}_i$ and $\bar{t}_i$ are the constants obtained from (23) by putting $\theta = \pi/2$, $\varphi = 0$, *i.e.*

$$(57) \quad \begin{cases} \bar{\mu}_i = \{-1, & 0, & 0, & 1\} \\ \bar{t}_i = \{ & 0, & -i, & -1, & 0\}. \end{cases}$$

To the first order we get for the tetrads (30) inside V

$$(58) \quad h_{(a)i} = \eta_{ai} + \frac{1}{2} \overset{(1)}{y}_{ai}$$

and the expression (31) for $T_i^k = t_i^k$ becomes

$$(59) \quad t_i^k = \frac{2 |\bar{c}'|^2}{\kappa} \bar{\mu}_i \bar{\mu}_k$$

where

$$(60) \quad \bar{c}' = \frac{d\bar{c}(\bar{u})}{d\bar{u}}$$

is the derivative of $\bar{c}(\bar{u})$ with respect to $\bar{u}$. Since the expression (6) for t_i^k is a homogeneous quadratic function of the first order derivatives of the tetrads the second order expression (59) for t_i^k is also obtained by introducing the first order tetrads (58) into (6).

The boundary condition for the tetrads formulated on p. 32 means for the region V that the $h_{(a)i}$ deviate from the corresponding tetrads in a completely empty space by quantities which are small of the first order and functions of $\bar{u}$ only. The permissible Lorentz rotations (14) are then, apart from constant finite rotations, those for which

$$(61) \quad \Omega_{(ab)} = \eta_{ab} + \bar{\omega}_{ab}(\bar{u}),$$

$\bar{\omega}_{ab}(\bar{u})$ any function of $\bar{u}$ which is small of the first order and antisymmetric in the indices a and b . The invariance of the second order expression for under such notations is shown in the same way as in the Appendix of ref. [4].

Further, as in Sect. 8 of ref. [4] it follows from the transformation properties of the energy-momentum complex that to the second order is invariant under all infinitesimal co-ordinate transformations

$$(62) \quad x'^i = \bar{x}^i + \xi^i$$

where $\xi^i(\bar{u})$ is a function of $\bar{u}$ only. By a transformation of this type the metric (54), (56) can be brought into its standard form, ref. [9]. Let $\chi(\bar{u})$, $\Phi(\bar{u})$ be two functions of $\bar{u}$ with the derivatives

$$(63) \quad \chi'(\bar{u}) = \bar{M}(\bar{u}), \quad \Phi'(\bar{u}) = \bar{\psi}(\bar{u}).$$

Choose

$$(64) \quad \xi^i(\bar{u}) = \chi(\bar{u})\bar{\mu}^i + \Re \{ \Phi(\bar{u})\bar{t}^i \}$$

with derivatives

$$(65) \quad \xi^i_{,k} = \frac{d\xi^i}{d\bar{u}} \frac{\partial u}{\partial \bar{x}^k} = M \bar{\mu}^i \bar{\mu}_k + \Re \{ \bar{\psi} t^i \} \bar{\mu}_k .$$

Then the components of the metric tensor in the new system are, by (56) and (65),

$$(66) \quad g'_{ik} = g_{ik} - \xi_{i,k} - \xi_{k,i} = \eta_{ik} + \overset{(1)'}{y}_{ik}$$

with

$$(67) \quad \overset{(1)'}{y}_{ik} = \overset{(1)}{y}_{ik} - \xi_{i,k} - \xi_{k,i} = 2\Re \{ \bar{c} \bar{t}_i \bar{t}_k \} .$$

Thus, in view of (57), g'_{ik} has the standard form where all the components of $\overset{(1)'}{y}_{ik}$ are zero except

$$(68) \quad \overset{(1)'}{y}_{22} = -\overset{(1)'}{y}_{33} = -2\Re \{ \bar{c} \} , \quad \overset{(1)'}{y}_{23} = \overset{(1)'}{y}_{32} = -2\Im \{ \bar{c} \} .$$

In this form the metric tensor depends only on the physically significant news function which is a function of $\bar{u}$ or of

$$(69) \quad u' \equiv x'^4 - x'^1 = \bar{x}^4 - \xi^4 - (\bar{x}^1 + \xi^1) = \bar{x}^4 - \bar{x}^1 = \bar{u} .$$

In the system $\bar{x}^i$ the components of the tetrads $h_{(a)i}$ in (58) are symmetrical in the indices a and i . In the new system (x'^i) defined by (62)-(64) the components $h'_{(a)i}$ are not symmetrical. However, by a suitable « gauge transformation » of the type (61), which does not change t_i^k , the symmetrical standard form may be obtained for the components $\check{h}'_{(a)i}$ of the rotated tetrads in the primed system of co-ordinates:

$$(70) \quad \check{h}'_{(a)i} = \eta_{ai} + \frac{1}{2} \overset{(1)'}{y}_{ai} .$$

It is easily seen that this is obtained by choosing

$$(71) \quad \bar{\omega}_{ab} = -\frac{1}{2} \Re \{ \bar{\psi} (\bar{\mu}_a \bar{t}_b - \bar{t}_a \bar{\mu}_b) \} .$$

For a sandwich wave where $\bar{c}'(\bar{u})$ vanishes outside a certain interval $\bar{u}_1 < \bar{u} < \bar{u}_2$ the momentum and energy per unit area in the $(\bar{y}\bar{z})$ -plane of the system $(\bar{x}^i)$ is by (57), (59)

$$(72) \quad \bar{p}_i = \int t_i^4 d\bar{x} = -\frac{2}{\pi} \bar{\mu}_i \int_{\bar{u}_1}^{\bar{u}_2} |\bar{c}'(\bar{u})|^2 d\bar{u} .$$

Since t_i^k is a tensor density under linear transformations the four-momentum $\bar{p}_i$ of this wave packet transforms as a 4-vector Lorentz

transformations of the co-ordinates. The invariant norm of this vector vanishes since $\bar{\mu}_i \bar{\mu}^i = 0$:

$$(73) \quad \bar{p}_i \bar{p}^i = 0 .$$

Therefore, as regard energy and momentum the packet of gravitational radiation is closely analogous to a packet of electromagnetic radiation. In the systems of co-ordinates $(\bar{x}^i)$ and (x'^i) Einstein's energy-momentum complex gives the same results as the complex T_i^k . On account of the Bauer difficulty this is in general not so in other systems of co-ordinates, obtained by more general infinitesimal transformations (62), where the Einstein expression may lead to meaningless results.

The virtue of the new formulation by means of tetrads is that it allows to account consistently for the measurable changes in energy and momentum of physical systems in complete accordance with the general principle of relativity. Besides giving the means to decide in which systems of co-ordinates Einstein's energy-momentum complex can be applied safely, it also clarifies certain dark points in the old formulation. For instance in the new formulation it is possible to *prove* that the total four-momentum of a microscopic nonradiative system, *i.e.* of a «particle», transforms as a 4-vector under arbitrary space-time transformations which lead from an inertial Lorentzian system of co-ordinates to arbitrarily accelerated systems of reference. This has always been *assumed* to be the case but it could not actually be proved since Einstein's energy-momentum complex gives meaningless results for the four-momentum in accelerated frames of reference. In this way the new formulation constitutes a certain rounding off of Einstein's beautiful theory of gravitation, which in many respects can be regarded as the last stage of a development started by Galileo 400 years ago.

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INTERVENTI E DISCUSSIONI

— H. BONDI:

The significance of an expression for the energy and momentum of a field depends in my view entirely on whether ideal experiments can be devised in which this energy is converted into a locally useful form of energy. This requirement seems to me to be far more important than whether the expression satisfies the general principle of relativity, which, following Fock, I regard as virtually empty.

Unhappily, the only really convincing measure of mass we have in general relativity is the effective Schwarzschild mass, available only for isolated systems in asymptotically flat space. Until we have physically significant ideal experiments to measure energy and momentum locally (perhaps following the tentative receiver model in my 1962 paper), the full potentialities of Møller's expressions can hardly be appreciated.

— C. MØLLER:

I do not agree with the statement that the general principle of relativity is empty. The requirement that all relations between measurable physical quantities (including of course the gravitational field variables) must have the same form in any system of space-time co-ordinates seems to me to be a meaningful and extremely useful working principle in a situation where our main difficulty is an «*embarass de richesse*». On the other hand, I entirely agree with Bondi that the next important question to be investigated is the absorption of gravitational waves. For this we would need to know the asymptotic form of the solutions of Einstein's field equations corresponding to a situation where an ingoing plane-gravitational wave is absorbed by a matter system, a problem which is even somewhat difficult to formulate properly.

— H. J. TREDER:

I think that we have an important difference between special and general relativity in the meaning of energy-momentum. The ideas of energy and so on are resulting from the special relativity (or from the classical physics), and I think that we have not good causes for the hope that we can find the same qualities in general relativity.

In special relativity the space-time possesses 10 symmetries and the 10 generators of the Lie-group are corresponding with the energy-momentum-vector and the angular-momentum. The space-time of general relativity does not possess symmetries in the general case. Therefore, general relativity does not possess generators corresponding with energy, etc. Of course, for the special spaces — for example the Schwarzschild space — in which a time-like Killing-vector exists, it is meaningful to speak from energy.

In general relativity the Noether-theorem gives 10 identities for qualities which are in some sense analogue to the energy-momentum etc. But these qualities are not homologies for energy etc., but only analogies.

In this sense I think that the complex of Prof. Møller is the complex which possesses the most analogies to the energy-momentum of special relativity, but the complex of Prof. Møller can't be the homolog of energy-momentum.

— C. MØLLER:

I do not think that I disagree with you, only I believe that we have now a consistent expression for the total energy and momentum for *any* insular system, not only for the static Schwarzschild case.

— M. A. TONNELAT:

The complex T_i^k is not invariant under the group of rotations except if the quantities Ω are constants. From this fact the energy T_4^4 is not, for the Engineer, a measurable quantity.

So may I conclude that there is no sense to ask if T_4^4 is positive defined or not?

— C. MØLLER:

From the point of view of my paper the answer is *yes*.

— A. MERCIER:

Vous interprétez l'éq. (14) à l'aide d'une invariance de jauge. Mais qu'est-ce qu'une transformation de jauge, dans son idée « la plus générale »? Et pourquoi exiger que les théories physiques, ou certains de leurs éléments, soient invariants lors de transformations dites de jauge?

— C. MØLLER:

If we only regard the total momentum and energy of a physical system as measurable but not their distribution throughout space, the transformations (14) in my paper are somewhat analogous to the gauge transformations of electromagnetic potentials, since the total four-momentum is invariant under (14) but not the complex T_i^k . I do not know what a gauge transformation « dans son idée la plus générale » is and therefore I am not able to answer your question.

— S. DESER:

General relativity is a theory which depends entirely on the metric, and *not* on all the sixteen vierbein components. This fact points out again the difficulties with even as nice a *local* T_i^k expression as Møller's, which depends intrinsically on the vierbeins. It is an interesting fact, however, that if one formulates the Einstein equations in vierbein terms, they take on an especially simple form with a partial fixation of vierbein orientation (gauge), corresponding to setting the time vierbein axes parallel to the time coordinate of the global coordinate frame used. This has recently been exploited by Schwinger, for example.

Now we have shown (*Phys. Lett.*, October 1963) that, precisely in this « time-gauge », the Møller four-momentum agrees with all the usual metric expressions (Einstein's, etc.) with suitable boundary conditions at infinity. This is a very satisfactory aspect of the Møller P_μ , and means that it is

independent of the remaining spatial vierbein orientation, and transforms appropriately under co-ordinate transformations which vanish asymptotically. But these results bring out again the fact that it is not physically desirable, either in special or in general relativity, to have invariance of P_μ with respect to generalized co-ordinates, that are *not* asymptotically minkowskian. It is in fact very much implicit in Møller's assumptions and in the physical meaning of energy, that there exist asymptotically flat frames, so that only invariance with respect to co-ordinate transformations respecting the boundary conditions at infinity should be required physically.

— C. MØLLER:

I think we agree, still I regard it is an important feature that the P_i for a non-radiative system behave as a free four-vector under arbitrary space-time transformations due to the true tensor density character of the super-potential U_i^{kl} .

— A. PAPAPETROU:

I should like to recall that there is another possibility to make T_i^k generally covariant. It is connected with what Professor Rosen calls the bi-metric interpretation of general relativity; let define in the space-time a «flat-space metric» $\gamma_{\mu\nu}$, i.e. a tensor field for which one has firstly $\det \gamma_{\mu\nu} > 0$ everywhere and then, if one assumes it formally to represent a metric and calculates its Christoffel symbols ${}^* \Gamma_{\mu\nu}^\alpha$ curvature tensor ${}^* R_{\lambda\mu\nu}^\alpha$, one finds ${}^* R_{\lambda\mu\nu}^\alpha = 0$ everywhere. It will then be sufficient to replace in the expressions for T_k^i the Christoffel symbols $\Gamma_{\mu\nu}^\alpha$ by the differences $\Gamma_{\mu\nu}^\alpha - {}^* \Gamma_{\mu\nu}^\alpha$ to obtain a covariant conservation law. Of course, one will have to impose on $\gamma_{\mu\nu}$ the condition $g_{\mu\nu} - \gamma_{\mu\nu} \rightarrow 0$ at spatial infinity in order to obtain physically useful results. One sees immediately that in this way the Bauer difficulty has been eliminated.

— C. MØLLER:

The idea of a bimetric formulation of general relativity forwarded by Papapetrou and by Rosen is very interesting. However, if it turns out that the distribution of energy and momentum are not measurable quantities but only the total energy and momentum, it seems that this theory gives us too much. This may be connected with the fact that the determination of the flat-space metric $\gamma_{\mu\nu}$ seems rather arbitrary since measurements with natural measuring sticks and clocks give the metric $g_{\mu\nu}$ and not $\gamma_{\mu\nu}$.

— E. NEWMAN:

I wish to point out that there is an alternate definition of an energy-momentum «vector» P_i , other than the one which was obtained by Prof. Møller from the Bondi *et al.* metric. If one adds to the Bondi-Møller P_i a specific quantity which is quadratic in the «news», one obtains a new $\hat{P}_i$, which also satisfies conservation laws similar to those satisfied by P_i but has in addition the advantage that the new quantities turn out to be essentially the Weyl tensor (or Riemann tensor) and hence can be measured by the methods of geodesic deviation.

— P. BERGMANN:

Attempting to summarize the discussion so far, I should be tempted to say that, regardless of the technology preferred, energy and momentum are well-defined if corresponding directions can be fixed. Because of the availability of superpotentials ($u_i^{(b)}$) directions in the interior are immaterial; what matter are the displacement vectors (δx^a) at the edge. In Bondi-type radiative solutions, as well as in other asymptotically flat solutions, the vector field is sufficiently well defined at infinity that total energy and momentum form a free vector. In the n -body problem it may be impossible to assign such free-vectors unambiguously to the individual constituent particles, but this is no disaster, as the localization of energy and momentum may be doubtful in Lorentz-covariant theories as well.

— A. KOMAR:

1) I am in full agreement with Professor Treder's comment concerning the formal character of the analogies which employ the generators of isometries in order to obtain formal expressions for conservation laws in general relativity.

2) Throughout classical physics the precise localization of energy is immaterial and ambiguous. Thus, for two particles connected by a Hooke's law spring, as the spring is compressed the energy may be localized either as potential energy of the first particle, or as potential energy of the second particle, or by symmetry as half in the first particle and half in the second, or finally the energy can be regarded as localized in the compressed spring. (This last situation corresponds to the «field theoretic» viewpoint). For Professor Bondi's «engineer» all the matter is that the total energy, however localized, is conserved. (Similar comments can be made for a charged particle in an electromagnetic field).

3) The only physical theory for which the precise localization is critical is general relativity, for the energy (and momentum, etc.) is the source of the gravitational field. Thus the gravitational field produced by the compressed Hooke's law spring would depend on the manner in which the energy density is in fact localized. However we are in this case concerned with the localization of *nongravitational* energy. We should neither expect or desire to localize the *gravitational* energy in a unique fashion for use as a source of *gravitational* field.

4) Prof. Bondi's «engineer» would of course be interested in localizing the *radiant energy* of the gravitational field. Since radiation is essentially a nonanalytic phenomenon, we are therefore concerned with discontinuities in derivatives of metric across characteristic surfaces. The additional geometric structures which thereby become available (e.g. the null cones, the discontinuities, Γ_{jk}^i , etc.) will enable the construction of new conservation laws appropriate for the localization of *gravitational radiant energy*. Such a construction should of course agree at infinity with the expressions of Einstein, Møller, J. N. Goldberg and others.

— A. LICHTNEROWICZ:

J'aimerais poser au Professeur Møller deux questions:

1) L'hypothèse de l'existence, sur une variété, d'un champ régulier de tétrades est une hypothèse topologiquement très forte. Elle interdit pratiquement de faire de la cosmologie.

Il serait très agréable que la densité d'énergie soit invariante par des rotations locales purement spéciales. S'il en était ainsi cette densité ne dépendrait que d'un système de lignes de temps (systèmes qui existent toujours sur un espace-temps). En est-il ainsi?

2) S'il en était ainsi, il serait très intéressant de comparer vos expressions avec celles résultant soit du point de vue du Prof. Cattaneo, soit du point de vue du Prof. Rosen.

— C. MØLLER:

The local quantity T_4^4 is not invariant under purely spatial rotations of the tetrads.

— S. DESER:

With respect to Prof. Lichnerowicz's important point, it is only the integrated energy expression which is independent of the spatial vierbein orientation once a choice of time gauge is made. On the other hand, the energy density remains highly dependent on the spatial orientation even when the time axis is decided; this dependence must be regarded, as Lichnerowicz points out, as a difficulty of the local quantities.

[Alla discussione partecipò anche I. ROBINSON.]

Définition et rôle d'une énergie gravitationnelle dans les théories minkowskiennes du champ de gravitation.

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1. - Représentation de l'impulsion-énergie gravitationnelle.

La notion « d'énergie » du champ s'est introduite dans la théorie des milieux continus dès l'époque de Faraday. On a supposé alors que les milieux devaient introduire des actions spécifiques, irréductibles à celles des sources.

Dès lors, le mouvement d'une particule d'épreuve attribué à l'influence des « forces » issues des sources peut se déduire d'une simple condition de conservation portant sur l'énergie totale du système champ-particule. Dans un système de coordonnées quelconques x^μ ($x^0 = x^1, x^2, x^3$; $x^0 = ct$), la conservation de cette impulsion-énergie totale $T_\mu{}^e$ s'écrit toujours:

$$(1) \quad \nabla_e T_\mu{}^e = 0$$

ou

$$(2) \quad \partial_e \sqrt{-g} T_\mu{}^e = \frac{\sqrt{-g}}{2} g^{\alpha\beta} \partial_\mu g_{\alpha\beta}$$

∇_e étant le symbole de dérivation covariante formé avec les symboles de Christoffel $\left\{ \begin{smallmatrix} e \\ \mu \nu \end{smallmatrix} \right\}$ relatifs eux-mêmes à une métrique $g_{\mu\nu}$. Jusqu'ici nous ne faisons aucune hypothèse sur la structure de l'espace.

L'interprétation de la condition (2) est néanmoins assez différente suivant qu'il s'agit d'une théorie de type phénoménologique (et par ce mot nous désignons des champs détachés de la géométrie) ou bien de la Relativité Générale.

a) Dans une *théorie phénoménologique*, T_μ^e groupe additivement les impulsions-énergies matérielles M_μ^e , électromagnétique τ_μ^e , et aussi gravitationnelle Σ_μ^e . D'autre part, il est toujours possible de rapporter (2) à un système minkowskien global. Aussi les conditions (2) peuvent toujours se ramener à l'expression suivante:

$$(3) \quad \partial_e (M_\mu^e + \tau_\mu^e + \Sigma_\mu^e) = 0.$$

S'il s'agit du champ électromagnétique $\varphi_{\mu\nu}$, on sait que τ_μ^e est le tenseur de Maxwell défini par:

$$(4) \quad \tau^{\mu\nu} \equiv -\frac{\delta \mathcal{L}_e}{\delta g_{\mu\nu}} \equiv -\frac{\partial \mathcal{L}_e}{\partial g_{\mu\nu}} + \partial_e \frac{\partial \mathcal{L}_e}{\partial (\partial_e g_{\mu\nu})} = -\frac{\partial \mathcal{L}_e}{\partial g_{\mu\nu}} \\ \left(\mathcal{L}_e = \frac{\sqrt{-g}}{4} g^{\mu e} g^{\nu \sigma} \varphi_{\mu\nu} \varphi_{e\sigma} \right)$$

dans une déduction lagrangienne classique.

On peut donc penser qu'une expression analogue:

$$(5) \quad \Sigma^{\mu\nu} \equiv -\frac{\delta \mathcal{L}_g}{\delta g_{\mu\nu}} \equiv -\frac{\partial \mathcal{L}_g}{\delta g_{\mu\nu}} + \partial_e \frac{\partial \mathcal{L}_g}{\partial (\partial_e g_{\mu\nu})}$$

doit représenter l'impulsion-énergie gravitationnelle dans le cadre d'une théorie phénoménologique du champ de gravitation. ($\mathcal{L}_g$, au contraire de $\mathcal{L}_e$, est explicitement fonction des $\partial_e g_{\mu\nu}$).

b) Au contraire, en *Relativité Générale*, le champ de gravitation est absorbé, comme les champs d'inertie, par la structure non euclidienne de l'espace-temps. Ceci entraîne que T_μ^e ne contient aucune contribution gravitationnelle et que (2) est susceptible d'une traduction purement *locale* dans un repérage minkowskien. Compte tenu des équations du champ, $T^{e\sigma} = (1/\chi) S^{e\sigma}$ on constate que le second membre de (2) peut se mettre sous la forme d'une divergence ordinaire:

$$(6) \quad \frac{\sqrt{-g}}{2} T^{e\sigma} \partial_\mu g_{e\sigma} \equiv -\partial_e \sqrt{-g} \theta_\mu^e$$

en désignant par θ_μ^e la forme canonique:

$$(7) \quad \sqrt{-g} \theta_\mu^e = \frac{1}{2\chi} \left(\frac{\partial \mathcal{G}}{\partial (\partial_e g_{\lambda\sigma})} \partial_\mu g_{\lambda\sigma} - \delta_\mu^e \mathcal{G} \right)$$

obtenu à partir d'une pseudo densité:

$$(8) \quad \mathcal{G} = \sqrt{-g} g^{\mu\nu} \left[\left\{ \begin{matrix} \varrho \\ \mu \nu \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ \varrho \sigma \end{matrix} \right\} - \left\{ \begin{matrix} \varrho \\ \mu \sigma \end{matrix} \right\} \left\{ \begin{matrix} \sigma \\ \varrho \nu \end{matrix} \right\} \right]$$

Aussi, la conservation (2) de l'impulsion totale s'écrit encore:

$$(9) \quad \partial_\varrho \sqrt{-g} (M_\mu{}^\varrho + \tau_\mu{}^\varrho + \Theta_\mu{}^\varrho) = 0.$$

Il semble donc que dans une théorie phénoménologique du champ de gravitation, l'impulsion énergie gravitationnelle soit définie per le tenseur eulérien $\Sigma_\mu{}^\varrho$ tandis qu'en Relativité Générale elle est liée au tenseur canonique $\theta_\mu{}^\varrho$. Les difficultés que suscitent l'emploi de $\theta_\mu{}^\varrho$ — qui, en Relativité Générale, n'est pas un tenseur — ont été analysées par C. MØLLER [13]. Nous n'y reviendrons pas ici.

Le rôle des tenseurs eulérien et canonique apparaît clairement dans les représentations de l'énergie gravitationnelle issues de la Relativité Générale. Par contre, il semble assez incertain dans la plupart des théories phénoménologiques de la gravitation. L'étude des identités de Noether-Weyl en fournit aisément le motif. A partir d'un lagrangien $\mathcal{L}(Y_{\mu\nu}, \partial_\varrho Y_{\mu\nu}, g_{\mu\nu}, \partial_\varrho g_{\mu\nu})$ fonction de la métrique (les $g_{\mu\nu}$ étant indépendant ou non) et éventuellement de potentiels phénoménologiques scalaire, vectoriel ou tensoriel (dont les composantes sont toujours des variables indépendantes), les dérivées eulériennes:

$$(10) \quad \sqrt{-g} A^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta Y_{\mu\nu}} \quad \sqrt{-g} t^{(\mu\nu)} = - \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}$$

et les grandeurs canoniques:

$$(11) \quad 2 \sqrt{-g} t_\mu{}^\varrho \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\varrho Y_{\lambda\sigma})} \partial_\mu Y_{\lambda\sigma} + \frac{\partial \mathcal{L}}{\partial(\partial_\varrho g_{\lambda\sigma})} \partial_\mu g_{\lambda\sigma} - \delta_\mu{}^\varrho \mathcal{L}$$

ou

$$(12) \quad 2 \sqrt{-g} t_\mu{}^\varrho \equiv \frac{\partial \mathcal{L}}{\partial(\nabla_\varrho Y_{\lambda\sigma})} \nabla_\mu Y_{\lambda\sigma} - \delta_\mu{}^\varrho \mathcal{L}$$

sont liés par les identités classiques:

$$(13) \quad \nabla_\varrho t_\mu{}^\varrho \equiv \nabla_\varrho (A^{\varrho\sigma} Y_{\mu\sigma}) - \frac{1}{2} A^{\varrho\sigma} \nabla_\mu Y_{\varrho\sigma}$$

et par l'une ou l'autre des conditions équivalentes:

$$(14) \quad t_{\mu}^{(m)g} - t_{\mu}^{(c)g} = A^{\nu} \Psi_{\mu\nu} + \frac{1}{\sqrt{-g}} \partial_{\sigma} \mathcal{B}_{\mu}^{\sigma g}$$

$$(15) \quad t_{\mu}^{(m)g} - t_{\mu}^{(c')g} = A^{\nu} \Psi_{\mu\nu} + \nabla_{\sigma} B_{\mu}^{\sigma g}$$

en posant:

$$(16) \quad \mathcal{B}_{\mu}^{\sigma g} \equiv \sqrt{-g} B_{\mu}^{\sigma g} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_{\sigma} \Psi_{\lambda\mu})} \Psi_{\lambda\mu} + \frac{\partial \mathcal{L}}{\partial (\partial_{\sigma} g_{\lambda\mu})} g_{\lambda\mu} \equiv -\mathcal{B}_{\mu}^{\sigma g}.$$

131 a) En *Relativité Générale*, les variables $\Psi_{\mu\nu}$ et, par suite, les quantités $A^{\mu\nu}$ disparaissent. Dès lors, les tenseurs eulérien et canonique satisfont toujours les identités:

$$(17) \quad \nabla_{\mu} t_{\mu}^{(m)g} \equiv 0 \quad \partial_{\mu} (t_{\mu}^{(m)g} - t_{\mu}^{(c)g}) \equiv 0.$$

Si le lagrangien — jusque là non explicité — est identifié à la pseudo densité $\mathcal{G}$ définie en (8), le tenseur eulérien $t^{\mu\nu}$ est le tenseur d'Einstein — $S^{\mu\nu} = -(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)$ et $t_{\mu}^{(c)g}$ coïncide avec l'expression θ_{μ}^g . Les équations du champ $S^{\mu\nu} = \chi M^{\mu\nu}$ entraînent alors:

$$(18) \quad \nabla_{\mu} M_{\mu}^g = 0 \quad \partial_{\mu} (M_{\mu}^g + \theta_{\mu}^g) = 0$$

le tenseur canonique est ainsi associé à l'impulsion-énergie du champ de gravitation.

b) Dans une *théorie phénoménologique*, l'impulsion énergie des sources est encore égale à la dérivée variationnelle du lagrangien par rapport aux variables indépendantes (c'est-à-dire par rapport au potentiel $\Psi_{\mu\nu}$). Mais ici, cette dérivée variationnelle est l'expression $A^{\mu\nu}$ définie en (14) et non pas le tenseur eulerien métrique.

Il en résulte que le tenseur eulerien métrique et le tenseur canonique semblent destinés l'un et l'autre à la description de l'impulsion-énergie du champ et que leur introduction est jugée, dans une large mesure, équivalente. On dit couramment que la « symétrisation » du tenseur canonique conduit à définir un tenseur identique, dans le vide, au tenseur eulérien d'impulsion énergie. En fait, sauf dans le cas d'un potentiel scalaire où les deux représentations coïncident, l'expression de l'impulsion-énergie du champ au moyen du tenseur canonique, fut-il symétrisé, est abusive dès qu'il

s'agit de l'appliquer à un mouvement de particules. On est donc amené, dès qu'il ne s'agit plus de la première approximation, à « reconstruire » le tenseur eulérien en ajoutant au tenseur canonique des termes supplémentaires plus ou moins arbitraires. (Dans la théorie de W. E. Thirring [20], « énergie de spin » qui n'est autre que l'expression (13) amputée de son dernier terme).

C'est ainsi qu'à l'exception de la théorie scalaire de Nördstrom [14], les théories euclidiennes de la gravitation n'ont pas réussi; en général, à rattacher les lois de champ et les lois du mouvement à une même déduction lagrangienne. L'énergie-impulsion du champ, que font intervenir explicitement ou implicitement (théorie de Birkhoff en loi de force) les équations du mouvement, *n'est pas celle qui correspond au lagrangien dont on peut déduire les équations du champ*. On peut dire encore que l'énergie totale correspondant à la déduction lagrangienne de la théorie n'est pas conservée (H. Weyl [24]).

2. - Energie gravitationnelle et mouvement des particules.

Ecrivons les éq. (3) en supposant la validité des équations du champ ($A^{\mu\nu} = \chi M^{\mu\nu}$). Compte tenu des identités, elles prennent la forme suivante:

$$(19) \quad \nabla_\rho (t_\mu^{\rho e} + M_\mu^e) \equiv \check{\nabla}_\rho \check{M}_\mu^e \quad (\text{modulo } A^{\mu\nu} = \chi M^{\mu\nu})$$

en posant:

$$(20) \quad \check{g}_{\mu\nu} = g_{\mu\nu} + \Psi_{\mu\nu} \quad \sqrt{-\check{g}} \check{M}^{\mu\nu} = \sqrt{-g} M^{\mu\nu}$$

et en désignant par $\check{\nabla}_\rho$ la dérivée covariante écrite avec les symboles de Christoffel formés eux-mêmes avec les $\check{g}_{\mu\nu}$.

Or, si $T^{\mu\nu}$ représente l'impulsion-énergie totale définie par le tenseur eulérien formé à partir du lagrangien total (champ + particule + interaction) la loi de conservation (A. Trautman [23])

$$(21) \quad A_\mu \equiv \nabla_\rho T_\mu^{\rho e} = 0$$

est équivalente à la condition:

$$(22) \quad \nabla_\rho (t_\mu^{\rho e} + M_\mu^e) = 0 \rightarrow \check{\nabla}_\rho \check{M}_\mu^e = 0$$

$t^{\mu\nu}$ définissant, comme en (10), l'impulsion-énergie eulérienne du champ.

Pour le montrer, il suffit d'appliquer à la déduction lagrangienne le paramétrage τ adapté, c'est-à-dire le paramétrage qui définit *identiquement* A_μ comme une intégrale première du mouvement ($\dot{z}^\mu A_\mu \equiv 0$, $z^\mu =$ coordonnée d'une particule, $\dot{z}^\mu = dz^\mu/d\tau$). En électrodynamique, A_μ , homogène et de degré 1, maintient l'indépendance des trajectoires par rapport au paramétrage et l'on choisit habituellement $d\tau = ds$. En gravitodynamique, A_μ homogène et de degré 2, entraîne $d\tau = d\tilde{s} = (\tilde{g}_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}}$. (S. Lederer et M. A. Tonnelat [9]).

La parenté entre l'expression $\tilde{\nabla}_e \tilde{M}_\mu^e = 0$ et le formalisme (18) de la Relativité Générale est évidente.

S'il s'agit d'une particule d'épreuve d'impulsion-énergie:

$$(23) \quad m_\mu^e = m_0 c^2 u_\mu u^e.$$

Les lois du mouvement et les conditions de conservation s'écrivent alors en adoptant le formalisme étoilé:

$$(24) \quad \tilde{u}^e \tilde{\nabla}_e \tilde{u}^\mu = 0 \quad \tilde{\nabla}_e (\tilde{m} \tilde{u}^e) = 0$$

avec:

$$(25) \quad \tilde{u}^\mu = \frac{u^\mu}{\omega} = \frac{dx^\mu}{d\tilde{s}} \quad \tilde{m} \sqrt{-\tilde{g}} = m \sqrt{-g} \omega^2$$

$$(26) \quad \omega^2 = \tilde{g}_{\mu\nu} u^\mu u^\nu = \frac{d\tilde{s}^2}{ds^2} \quad (\omega^2 = 1 + \Omega^2 = \tilde{g}_{\mu\nu} u^\mu u^\nu \Omega^2 = \Psi_{\mu\nu} u^\mu u^\nu) (*).$$

Bien entendu, le formalisme étoilé représente ici une simple notation qui donne aux conditions de conservation et aux lois du mouvement (22) et (24) des formes très parentes de celles que l'on obtient en Relativité Générale.

Retranscrites dans l'espace physique minkowskien, les lois du mouvement prennent la forme suivante dans un repère adapté $g_{\mu\nu} = \eta_{\mu\nu} = (-\delta_{\mu 0}, \delta_{\mu 0})$

$$(27) \quad \frac{d}{ds} (\eta_{\mu\nu} + \Psi_{\mu\nu}) \frac{u^\nu}{\omega} = \frac{1}{2\omega} u^\alpha u^\sigma \partial_\mu \Psi_{\sigma\alpha}$$

$$(28) \quad \partial_e (m' \sqrt{-g} u^e) = 0 \quad \text{avec } m' = m\omega.$$

La masse conservative est ainsi la masse m' .

(*) Des notations de ce type s'introduisent naturellement dans les théories de ce type et se trouvent avec quelques variantes dans les travaux de Rosen [18] et de W. E. Thirring [21]. Elles sont indépendantes de l'explicitation lagrangienne. Ajoutons que cette théorie qui postule des équations de champ strictement linéaires mais essentiellement différentes de l'approximation linéaire de la Relativité Générale s'écarte radicalement des principes des deux théories précitées.

En notations étoilées, les équations du mouvement (24) sont formellement identiques à celles de la Relativité Générale. Par contre, les équations du champ, strictement linéaires, sont nécessairement différentes de l'approximation linéaire de la loi d'Einstein $S^{\mu\nu} = \chi M^{\mu\nu}$. Celle-ci peut s'écrire

$$(29) \quad \square \Psi_{\mu\nu} - (\partial_\mu \Gamma_\nu + \partial_\nu \Gamma_\mu) + \partial_\mu \partial_\nu \Psi + \eta_{\mu\nu} (\partial^\sigma \Gamma_\sigma - \square \Psi) \sim -2\chi M_{\mu\nu},$$

en posant:

$$(30) \quad g_{\mu\nu} = \eta_{\mu\nu} + \Psi_{\mu\nu}, \quad \Gamma_\mu = \partial^\sigma \Psi_{\mu\sigma}, \\ \Psi = \eta^{\mu\nu} \Psi_{\mu\nu}, \quad \chi = \frac{8\pi G}{c^4} \quad (\Psi^2, (\partial_\sigma \Psi)^2 \ll 1)$$

et, comme on le sait, reste invariante dans un changement de jauge:

$$(31) \quad \Psi_{\mu\nu} \rightarrow \Psi'_{\mu\nu} = \Psi_{\mu\nu} + \partial_\mu \chi_\nu + \partial_\nu \chi_\mu.$$

Cette propriété permet d'imposer l'approximation dite « isotherme »:

$$(32) \quad \Gamma_\mu = \frac{a}{2} \partial_\mu \Psi \quad a = \text{constante donnée}$$

qui entraîne toujours quel que soit a :

$$(33) \quad \partial^\mu M_{\mu\nu} = 0$$

c'est-à-dire la prévision de géodésiques rectilignes. Cette condition, naturelle tant qu'elle est approchée, devient inacceptable s'il s'agit d'une relation rigoureuse.

Les résultats précédemment acquis ont permis de discuter la forme générale mais linéaire des équations du champ (S. Mavridès [11]). On constate que (29), approximation linéaire de la Relativité Générale, se présente comme un cas singulier que doit exclure une théorie strictement euclidienne. D'autre part on peut montrer qu'il est possible d'attribuer au lagrangien la forme simple

$$(34) \quad \mathcal{L} = \partial_\sigma \Psi_{\mu\nu} \partial^\sigma \Psi^{\mu\nu} - \partial_\sigma \Psi \partial^\sigma \Psi$$

qui, associée au terme usuel d'interaction $2\chi \Psi^{\mu\nu} M_{\mu\nu}$, conduit aux équations du champ

$$\square (\Psi_{\mu\nu} - \eta_{\mu\nu} \Psi) = -\chi M_{\mu\nu}.$$

Celles ci font intervenir, comme en Relativité Générale, une seule constante arbitraire κ .

L'approximation newtonienne déduite de (3) en fixe d'ailleurs immédiatement la valeur:

$$(36) \quad \kappa = -\frac{12\pi G}{c^4}.$$

3. - La précession des périhélies des planètes et des satellites sous l'influence du champ de gravitation créée par un corps central en rotation.

Le test essentiel de la validité d'une théorie de la gravitation consiste évidemment dans la détermination de trajectoires d'une planète ou d'un satellite. Ce test fait intervenir en effet les lois du champ et du mouvement tandis que les lois relatives aux trajectoires et à la fréquence de la lumière impliquent seulement la donnée des équations du champ. Une première approximation des lois du mouvement est celle qui conduit aux trajectoires newtoniennes. La seconde approximation permet de déterminer la précession des périhélies dans le champ d'un corps central statique (précession de Schwarzschild dans le cas de la Relativité Générale). Enfin, la troisième approximation conduit à tenir compte d'un éventuel mouvement de la rotation du corps central.

Dans le cas de la Relativité Générale, on prévoit ainsi, outre la précession que produit un champ statique:

$$(37) \quad \Delta \bar{\omega} = 6\pi \frac{GM}{h^2} \frac{GM}{c^2} \quad \text{radian par révolution,}$$

M = masse du corps central, G constante newtonienne,

$$h^2 = GMa(1 - e^2) \quad (a = \frac{1}{2} \text{ grand axe, } e = \text{excentricité}),$$

une précession séculaire du nœud et du périhélie:

$$(38) \quad 2 \Delta \Omega = -\frac{\Delta \bar{\omega}}{\cos i} = \frac{\pi^2 J R^2}{9 c^2 \tau T^2} \quad \text{secondes d'arc}$$

sans l'adjonction d'aucune autre perturbation

Ω et $\bar{\omega}$ = longitudes du nœud et du péricentre;

τ = période de rotation du corps central en jours;

T = période de révolution du satellite en jours;

i = inclinaison; R rayon du corps central;

J = nombre de jours de l'année sidérale.

Le calcul résulte de la méthode utilisée par Lense et Thirring [10], [20], [21] puis modifiée par N. Kalitzin [8] pour rapporter la précession des péricentres au plan de la trajectoire du satellite (cos i remplaçant alors $1 - 2 \sin^2(i/2)$).

Au contraire, en appliquant les méthodes de perturbation à la théorie newtonienne on obtient, à la seconde approximation la précession classique des périhélies (à l'exception de l'avance séculaire résiduelle de $43''$ d'arc) mais aucune précession des périhélies ou des nœuds ne correspond, en troisième approximation, à l'influence de la rotation du corps central. Autrement dit, la précession séculaire de rotation $\Delta\bar{\omega}_{\text{rot}}$ est nulle dans une théorie strictement newtonienne.

Plus récemment, V. I. Pustovoit [15] a calculé l'influence de la rotation du corps central dans les théories minkowskiennes de F. J. Belinfante et J. C. Swihart [2] et dans la théorie de Birkhoff [3].

Dans cette dernière l'effet spécifique dû à la rotation du corps central est les $3/8$ de celui de la Relativité Générale mais nous avons signalé des difficultés évidentes des principes de la théorie (non-conservation de l'énergie). Par contre, dans la tentative de F. J. Belinfante et de J. C. Swihart, la rotation du corps central entraîne une modification séculaire de la longueur du grand axe et de l'excentricité des orbites, conclusion difficilement admissible.

En cherchant alors les solutions de *seconde approximation* de l'équation du champ:

$$(39) \quad \square(\Psi_{\mu\nu} - \eta_{\mu\nu}\Psi) = -\frac{12\pi G}{c^4} M_{\mu\nu}$$

solutions valables dans un cas statique à symétrie sphérique et portant ces solutions dans les équations du mouvement (27) écrites à la même approximation on obtient exactement l'avance classique (S. Mavridès [11]):

$$(40) \quad \Delta\bar{\omega} = 6\pi \cdot \frac{GM}{h^2} \frac{GM}{c^2},$$

qui correspond à la valeur séculaire de $43''$ dans le cas de Mercure.

L'approximation suivante fait alors intervenir la rotation propre du corps central. En désignant par la notation ' la dérivée d/ds et par I le moment d'inertie du corps central, les équations du mouvement au

troisième ordre d'approximation sont alors les suivantes :

$$(41) \quad \mathbf{r}'' \left(1 + \frac{GM}{c^2 r} \right) = - \frac{GM}{2c^2 r^2} \left[\frac{2\mathbf{r}}{r} + \frac{3\mathbf{r}'}{r} \wedge [\mathbf{r} \wedge \mathbf{r}'] - \mathbf{r}' \cdot \mathbf{r}' \right] + \\ + \frac{3}{2} \frac{GI}{c^3 r^3} \left[\mathbf{r}' \wedge \boldsymbol{\omega} + \frac{3}{2} (\mathbf{r} \cdot \boldsymbol{\omega}) [\mathbf{r} \wedge \mathbf{r}'] \right].$$

On a posé :

$$\mathbf{r} = (x, y, z), \quad \mathbf{r}' = \frac{d\mathbf{r}}{ds} = (x', y', z'), \quad \mathbf{r}'' = \frac{d^2\mathbf{r}}{ds^2} = (x'', y'', z''), \quad \boldsymbol{\omega} = (0, 0, \omega).$$

A partir de ces expressions, on peut alors prévoir un retard séculaire des nœuds ($\Delta\Omega$) et des péricentres ($\Delta\bar{\omega}$) égal à :

$$(42) \quad 2 \Delta\Omega = - \frac{\Delta\bar{\omega}_{\text{rot}}}{\cos i} = \frac{M^2 J R^2}{12 c^2 \tau T^2} \text{ secondes d'arc}$$

retard égal, par conséquent, aux 3/4 de la valeur prévue par la Relativité Générale.

L'effet ainsi prévu est négligeable tant qu'il s'agit des grosses planètes. En effet la vitesse de rotation du soleil est relativement faible et le retard qui serait introduit même dans le cas le plus favorable — celui de la planète Mercure — est extrêmement faible ($< 0''01$).

Le retard prévu par la Relativité Générale avait été calculé dans le cas du 5^e satellite de Jupiter. La grande vitesse de rotation propre de Jupiter permet de prévoir un retard spécifique du périjove — $3'46''$, qui se retrancherait de l'avance $36'37''$ due au champ statique.

Ici la valeur du retard spécifique est seulement [12] :

$$\Delta\bar{\omega}_{\text{rot}} = - 2'49''$$

et reste évidemment beaucoup trop faible pour servir de test à la théorie et a fortiori pour déceler une différence entre les conclusions ainsi obtenues et celles de la Relativité Générale.

Il importe de noter que la théorie minkowskienne déduite des principes que nous venons d'exposer conduit, comme la Relativité Générale, à la prévision de trajectoires stationnaires sans modifications séculaires des éléments — grand axe et excentricité de l'ellipse.

Le cinquième satellite de Jupiter, Amalthée, (5^e dans l'ordre de la découverte) est le plus proche de Jupiter et, par conséquent, celui qui est le plus sensible aux effets de la rotation propre de l'astre central. Toutefois la trajectoire de ce satellite est extrêmement sensible à l'influence de l'aplatissement de Jupiter.

Des corrections moindres devraient être apportées aux satellites moins proches de Jupiter, c'est-à-dire au premier et au second satellite (dans l'ordre de la découverte) mis en évidence par GALILÉE en 1610 [6]. Pour ces

satellites les retards dûs à la rotation du corps central seraient seulement:

$$\text{pour Io} \quad \Delta\bar{\omega}_{\text{rot}} = -13'';$$

$$\text{pour Europe} \quad \Delta\bar{\omega}_{\text{rot}} = -3'',$$

d'après la théorie minkowskienne considérée ici.

Ces valeurs sont trop différemment accessibles pour permettre des discriminations systématiques. Il n'en reste pas moins vrai que les observateurs de 1610 ont ouvert un champ d'observation qui, sans être encore tout à fait exploitable ni exploité, contribuera peut-être, avec le perfectionnement des moyens techniques, à la définition systématique d'une impulsion-énergie gravitationnelle et à la confirmation de la Relativité Générale en tant que théorie non euclidienne du champ de gravitation.

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INTERVENTI E DISCUSSIONI

— W. B. BONNOR:

If one considers the planetary orbits of a spinning star, is there any difference between the theories of Einstein and Whitehead?

— M. A. TONNELAT:

I did not see such reasearch for a spinning star in the theory of Whitehead. It's interesting to consider this point, but my first preoccupation was to compare the results obtained by field theories with similar Lagrangian methods like general relativity and Minkowskian theories.

— P. HAVAS:

Guided by the same motivation as Mme Tonnelat, to better understand some features of the general theory of relativity, I carried out some calculations using a special relativistic theory of gravitation some years ago. (P. HAVAS: *Bull. Am. Phys. Soc.*, 6, 346 (1961)). The difficulty I was concerned with is connected with the problem of gravitational radiation. If this problem is approached via the equations of motion (including radiation reaction terms) obtained by a Lorentz-invariant approximation method (P. HAVAS: *Phys. Rev.*, 108, 1351 (1957); P. HAVAS and J. N. GOLDBERG: *Phys. Rev.*, 128, 398 (1962)), it is found that in the first nontrivial order these equations lead to an energy gain for a two-body system rather than the expected energy loss (S. F. SMITH and P. HAVAS: *Bull. Am. Phys. Soc.*, 5, 53 (1960) and *Phys. Rev.* (submitted Aug. 1964)). To understand the possible significance of this result it seemed worthwhile to approach the problem from a different angle, avoiding both the nonlinearity and any difficulty of interpretation of the general theory of relativity from the outset. Therefore the problem was investigated within the framework of Belinfante's first linear theory of gravitation (F. J. BELINFANTE: *Phys. Rev.*, 89, 914 (1953)), which is the most general linear special relativistic theory derivable from a Lagrangian, containing a number of adjustable parameters. The equations of motion of point masses including radiation reaction terms can be obtained by well-known special relativistic methods. They were applied to the two-body problem, and the dependence of the rate of change of the energy of this system on the adjustable parameters was determined. If these parameters are chosen such as to yield the Newtonian equations in the low-velocity limit and the correct advance of the perihelion in the next approximation, it was found that there is an energy *gain* of the system due to radiation effects. If instead they are chosen such as to yield an energy *loss* for all values of the eccentricity in an elliptic orbit, these values correspond to an advance of the perihelion of at most 5/42 of the correct amount.

The possibility of an energy gain in spite of retarded fields and outgoing waves is due to the fact that the canonical energy-momentum tensor of Belin-

fante's theory permits waves with an energy flow opposite to their direction of propagation.

Thus a special relativistic linear theory of gravitation leads to similar difficulties with gravitational radiation as general relativistic calculations which treat this radiation as a small effect superimposed on a flat background metric.

— M. A. TONNELAT:

1) For all that is concerned by radiation problems, I began calculations including radiation reaction terms. I deduced by a Lorentz-invariant approximation method the solutions but in very particular cases at this time. The calculations must be continued to obtain a significative answer.

2) Besides any question of radiation, the density of the field energy-momentum t^{00} is very complicated in this theory. In fact, the Eulerian energy tensor deduced from a Lagrangian density such as

$$\sqrt{-g} \nabla_e \psi_{\alpha\beta} \nabla^e \psi^{\alpha\beta}$$

depends explicite upon the first derivatives of the metric $\partial_e g_{\alpha\beta}$ contained in the ∇_e . (The Euclidean $g_{\alpha\beta}$ can be reduced to Minkowskian values $\eta_{\alpha\beta}$ but after variations only). So, the Eulerian expression

$$t^{(m)\mu\nu} = - \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} \neq - \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}$$

is not like the Maxwellian one, because the Maxwellian field is a curl without any $\partial_e g_{\mu\nu}$. From this fact result in gravitational case some terms like $\psi^e_\beta \partial^{(\nu} \partial^e \psi^{\mu)\beta}$ in the expression of energy-momentum $t^{\mu\nu}$. In spite of that it is possible to show that t^{00} is positive definite in the most important static cases.

3) For the simplification of this report I have not discussed the most general point of view for introducing such a theory. The form (34) postulated here has already taken in account the following effects (Mavridès [11]):

a) Relativistic advance of the perihelion (at the second order).

b) Relativistic deviation of light rays. It remains at this time, to obtain a satisfactory form for the energy of radiation *lost* by the system.

c) In my opinion a difficulty of the Belinfante theory consists of introducing 2 couplage constants in a theory with spin 2 and zero are not separated. A further difficulty is in obtaining secular variations of orbits (Pustovoit [15]). But the main difficulty of any Minkowskian theory based on the conservation of the canonical (and not Eulerian) energy-momentum tensor remains the impossibility of being deduced with coherence by a Lagrangian method (Trautman [24]). It is the essential point of view on which I intend to insist.

— S. DESER:

It seems to be a puzzling thing about the flat space representations of gravitation that they fell into two radically different classes. In each case, one abandons of course the equivalence principle *ab initio*, since a preferred

global inertial frame is assumed. The first is the linear phenomenological theory of the type you suggest, which maintains the preferred frame idea throughout. The second is the well-known procedure due to Papapetrou, Gupta and others, which also starts with a flat-space spin two theory and couples it to the total stress-tensor. To keep consistency one is then forced to generate an infinite series in powers of the spin two field, and (if this is done in one particular « correct » way) thereby obtains an expression identical to the Einstein's $G_{\mu\nu}$ expanded in powers of the metric. Now if the identification of the spin two field with $(g_{\mu\nu} - \eta_{\mu\nu})$ is made (presumably this is not necessary) one has reached the Einstein theory and with it the equivalence principle which was explicitly missing at the start. This fact appears paradoxical, even if one remembers that the identification is an extra assumption.

— M. A. TONNELAT:

The second point of view (successive approximations Papapetrou-Gupta) is in relation with the following.

We ask for field equations such as $\overset{0}{A}{}^{\mu\nu} + \overset{1}{A}{}^{\mu\nu} + \dots = \chi \mathcal{M}{}^{\mu\nu}$, with

$$\overset{0}{A}{}^{\mu\nu} = \frac{\delta \overset{0}{\mathcal{L}}}{\delta \psi_{\mu\nu}}, \quad \overset{1}{A}{}^{\mu\nu} = \frac{\delta \overset{1}{\mathcal{L}}}{\delta \psi_{\mu\nu}}, \quad [\overset{0}{\mathcal{L}}(\eta_{\alpha\beta}, \partial_\alpha \psi_{\alpha\beta}), \overset{1}{\mathcal{L}}(\eta_{\alpha\beta}, \psi_{\alpha\beta}, \partial_\alpha \psi_{\alpha\beta})]$$

$\overset{0}{\mathcal{L}}$ is a quadratic function (invariant density) of the phenomenological first derivatives $\partial_\alpha \psi_{\mu\nu}$. They chose a Minkowskian metric and they have terms with different coefficients $\alpha, \beta, \dots$. By postulating a gauge invariance property, coefficients $\alpha, \beta, \dots$ must be chosen so that $\overset{0}{A}{}^{\mu\nu}$ and $\overset{0}{\mathcal{L}}$ are the first approximation of general relativity.

Then, it is possible to calculate the energy-momentum tensor $\overset{0}{l}{}^{\mu\nu}$ from known $\overset{0}{\mathcal{L}}$ and to put

$$\overset{0}{\mathcal{L}} \rightarrow \overset{0}{t}{}^{\mu\nu} = \frac{\delta \overset{0}{\mathcal{L}}}{\delta \psi_{\mu\nu}} = \overset{0}{A}{}^{\mu\nu}, \text{ etc. .}$$

It is possible by identities (13) and (14) to establish an identity between $\overset{*}{\nabla}_e \overset{0}{t}_\mu{}^{e*}$ and $\overset{*}{\nabla}_e \overset{0}{A}_\mu{}^{e*}$ that is to say between the Gupta's quantities $\overset{*}{\nabla}_e \overset{1}{A}_\mu{}^{e*}$ and $\overset{*}{\nabla}_e \overset{0}{A}_\mu{}^{e*}$ ($g_{\mu\nu} = \eta_{\mu\nu} + \psi_{\mu\nu}$).

Then, we obtain finally an identity such as the following:

$$\overset{*}{\nabla}_e (A_\mu{}^{e*}) = \overset{*}{\nabla}_e (\overset{(0)}{A}_\mu{}^{e*} + \overset{(1)}{A}_\mu{}^{e*} + \dots + \overset{(n)}{A}_\mu{}^{e*}) \equiv 0.$$

It is easy to conclude: it is well known that such an expression $A^{\mu\nu}$, symmetrical tensor, a function of $g_{\mu\nu}$ and of its two first derivatives and whose divergency is identically equal to zero, is Einstein's tensor:

$$A^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \overset{*}{l} g^{\mu\nu} \text{ (Cartan's theorem) .}$$

Such methods with *gauge invariance postulate* must finally lead to general relativity.

On the contrary the first point of view is not at all concerned in the same manner with gauge invariance.

The second point of view can be related with the following.

We search field equations (linear or not) but connected with *metric and not with some phenomenological quantity*.

Such field equation:

$$E_{\mu\nu}(g_{\alpha\beta}, \partial_e g_{\alpha\beta}, \partial_{e\sigma}^2 g_{\alpha\beta}) = \chi M_{\alpha\beta}$$

can be deduced by a Lagrangian method from an invariant density. Then, they can be associated with equations of motion:

$$A_\mu = \frac{\delta \mathcal{L}}{\delta \zeta^\mu} = 0,$$

equivalent to:

$$\nabla_e t_\mu{}^e \equiv \nabla_e (M_\mu{}^e + t_\mu{}^e) = 0, \quad t^{\mu\nu} = -\frac{\delta \mathcal{L}}{\delta g_{\mu\nu}}, \quad \text{if } E_{\mu\nu} = t_{\mu\nu}.$$

Then, by classical identities,

$$\nabla_e t_\mu{}^e \equiv 0 \rightarrow \nabla_e{}^e M_\mu = 0$$

for any Lagrangian density.

— H. BONDI:

The measurements of the advance of the perihelion of the Earth and of Mars is of vital importance for general relativity. Just as in the case of artificial satellites, where deviations of the Earth field from spherical symmetry are of dominant importance, so in the case of Mercury the observed perihelion advance can always be ascribed to internal properties of the Sun rather than to the general relativity effect, but this relativity effect falls off far more slowly with distance than other possible causes affecting the orbit of Mercury.

— M. A. TONNELAT:

I agree. The recent researches about the spherical symmetry of the sun are very important but at this time they are unable to give any advantage to general relativity or to the Minkowskian theory.

[Alla discussione partecipò anche A. PAPAPETROU.]

Energy and Gravitational Waves in Bi-Metric Relativity Theory.

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1. - Bi-metric relativity theory.

In the general theory of relativity one considers space-time to be a four-dimensional Riemannian space, the geometry of which is described by means of a metric tensor $g_{\mu\nu}$. The fundamental idea of bi-metric relativity is that, alongside of this metric tensor, one introduces everywhere a second metric tensor $\gamma_{\mu\nu}$ corresponding to flat space, *i.e.*, one for which the Riemann-Cristoffel curvature tensor vanishes [1, 2, 3].

The presence of the flat-space metric turns out to be convenient from the formalistic standpoint. Thus, certain quantities which otherwise would have complicated transformation properties now become tensors. In particular, it is possible to define an energy-momentum density tensor for the gravitational field, something which is not possible in the usual form of general relativity. One can therefore regard bi-metric relativity simply as general relativity theory with a modification of the formalism, but without any conceptual changes.

On the other hand, one can introduce conceptual changes. One can give a new interpretation to bi-metric relativity by regarding $\gamma_{\mu\nu}$ as the metric tensor describing the geometry of space-time (this now being flat), while $g_{\mu\nu}$ is considered as a gravitational potential associated with the dynamics of material bodies, rather than with geometry.

Either one of these points of view appears to be tenable. In looking for order in nature, one must choose a conceptual framework within which to describe the ordering. Different conceptual frameworks are sometimes possible, the choice among them being made on the basis of convenience or of taste.

In general relativity the interval between two neighbouring events is given by

$$(1) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

In principle, this interval is what is determined by means of measuring

rods and clocks, on the assumption that the length of the measuring rod and the rate of the clock are invariant quantities, not affected by the gravitational field. The general form of this interval corresponds to a curved Riemannian space.

The alternative point of view is that the interval is given by

$$(2) \quad d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu.$$

This can be considered to be the interval as measured by rods and clocks, where however we now assume that these are affected by gravitation, and we therefore make appropriate corrections. The corrections are such that the form of the interval turns out to correspond to a flat four-dimensional space.

For the purpose of the present discussion we will simply assume that at each point of space-time we have two metric tensors, $\gamma_{\mu\nu}$ and $g_{\mu\nu}$, without specifying which of the two is associated with the geometry of the physical space-time. The tensor $\gamma_{\mu\nu}$ has a vanishing curvature tensor, so that its components are uniquely determined by the nature of the co-ordinate system with which we are working. The tensor $g_{\mu\nu}$ is determined in the given co-ordinate system by the field equation and the boundary conditions. The flat space associated with $\gamma_{\mu\nu}$ will be referred to as the γ -space, while the Riemannian space having the geometry described by $g_{\mu\nu}$ will be called the g -space.

It might be pointed out that in the flat γ -space, where one can compare unambiguously the directions of two vectors at a distance, it is possible to set up quadruples of axes at various points so that corresponding axes are all parallel to one another. In this way one can build up the formalism of « distant parallelism » [4, 5]. The large number of elements in this formalism might perhaps enable one to give a geometrical description of fields other than gravitation. However, if we restrict ourselves to gravitation, the quadruples do not appear necessary. We will therefore deal here only with the two metric tensors.

We can now define two kinds of covariant differentiation, that based on $g_{\mu\nu}$ (g -differentiation, denoted by a semicolon) and that based on $\gamma_{\mu\nu}$ (γ -differentiation, denoted by a comma). Ordinary partial differentiation with respect to x^μ , when needed, will be denoted by ∂_μ . Let us denote the Christoffel three-index symbols formed from $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ by $\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\}$ and $\Gamma_{\mu\nu}^\lambda$ respectively. One readily verifies that

$$(3) \quad \left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\} = \Delta_{\mu\nu}^\lambda + \Gamma_{\mu\nu}^\lambda,$$

where $\Delta_{\mu\nu}^\lambda$ is a tensor having the same form as $\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\}$, but with partial deriv-

atives replaced by γ -derivatives. In the same way one finds the relation

$$(4) \quad R_{\mu\nu\sigma}^{\lambda} = K_{\mu\nu\sigma}^{\lambda} + P_{\mu\nu\sigma}^{\lambda},$$

where $R_{\mu\nu\sigma}^{\lambda}$ and $P_{\mu\nu\sigma}^{\lambda}$ are the Riemann-Christoffel tensors formed from $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ respectively, and $K_{\mu\nu\sigma}^{\lambda}$ has the same form as $R_{\mu\nu\sigma}^{\lambda}$, but with partial derivatives replaced by γ -derivatives.

Since the γ -space was assumed to be flat, we have

$$(5) \quad P_{\mu\nu\sigma}^{\lambda} = 0.$$

Hence we can write

$$(6) \quad R_{\mu\nu\sigma}^{\lambda} = -\Delta_{\mu\nu,\sigma}^{\lambda} + \Delta_{\mu\sigma,\nu}^{\lambda} + \Delta_{\alpha\nu}^{\lambda} \Delta_{\mu\sigma}^{\alpha} - \Delta_{\alpha\sigma}^{\lambda} \Delta_{\mu\nu}^{\alpha}.$$

The Ricci tensor is then given by

$$(7) \quad R_{\mu\nu} = R_{\mu\nu}^{\beta}{}_{\beta} = -\Delta_{\mu\nu,\beta}^{\beta} + \Delta_{\beta\mu,\nu}^{\beta} - \Delta_{\beta\alpha}^{\beta} \Delta_{\mu\nu}^{\alpha} + \Delta_{\beta\mu}^{\alpha} \Delta_{\alpha\nu}^{\beta}.$$

In these and other examples one finds that, in general, one can rewrite the equations of general relativity theory so that $\left\{ \begin{smallmatrix} \lambda \\ \mu \nu \end{smallmatrix} \right\}$ is replaced by $\Delta_{\mu\nu}^{\lambda}$, ordinary derivatives by γ -derivatives, $(-g)^{\frac{1}{2}}$ by

$$(8) \quad \chi = (g/\gamma)^{\frac{1}{2}},$$

where g and γ are the determinants of $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ respectively and, in integrations, $d\tau$ by $(-\gamma)^{\frac{1}{2}} d\tau$ where $d\tau = dx^1 dx^2 dx^3 dx^4$. One returns to the usual expressions of Einstein general relativity by setting $\gamma_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is the metric tensor of special relativity theory.

It should be remarked that the above relations, in particular (3) and (4), are rather striking. They lend plausibility to the idea of introducing the γ -metric into the formalism.

2. - Field equations and energy.

One of the beautiful features of the Einstein general relativity theory is the fact that, once one has accepted its fundamental postulates, there is very little arbitrariness in it; in particular, one arrives at a nearly unique set of field equations. In the present case, if we started from the beginning with the additional metric $\gamma_{\mu\nu}$ at our disposal and looked for possible field equations, we would be faced with a large number of possi-

bilities. However, the standpoint adopted here is that we do not start from the beginning, but rather take over the basic formalism of the general relativity theory. The field equations chosen are therefore those of Einstein. On the one hand, this choice can be justified by the great success of the general relativity theory. On the other hand, one can present some theoretical arguments in support of this choice, as will be seen.

The field equations are therefore taken in the form

$$(9) \quad G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu},$$

where $T_{\mu\nu}$ is the energy-momentum density tensor of matter and of all fields other than gravitation. Since there exist the identities

$$(10) \quad G^{\mu\nu}_{;\nu} \equiv 0,$$

these equations are not sufficient to determine the $g_{\mu\nu}$ for given $\gamma_{\mu\nu}$, and four more conditions are needed. It is proposed to take for this purpose

$$(11) \quad \bar{g}^{\mu\nu}_{,;\nu} = 0,$$

where $\bar{g}^{\mu\nu} = \chi g^{\mu\nu}$. These conditions are similar to the De Donder conditions and reduce to them in the case $\gamma_{\mu\nu} = \eta_{\mu\nu}$. The field equations, together with these conditions, and the boundary conditions at infinity, namely, the requirement that

$$(12) \quad g_{\mu\nu} = \gamma_{\mu\nu} \quad (\infty),$$

and the specification of the asymptotic behavior (*e.g.*, that corresponding to outgoing waves) should serve to fix the $g_{\mu\nu}$ for a given $T_{\mu\nu}$.

The field equations can be derived from a variational principle. Let us write

$$(13) \quad L = L_m + L_g,$$

where L_m describes the matter and non-gravitational fields and is such that

$$(14) \quad \frac{\delta L}{\delta g_{\mu\nu}} \equiv \frac{\partial L_m}{\partial g_{\mu\nu}} - \partial_\alpha \frac{\partial L_m}{\partial (\partial_\alpha g_{\mu\nu})} = \chi T^{\mu\nu},$$

and

$$(15) \quad L_g = \frac{1}{8\pi} \bar{g}^{\mu\nu} (\Delta_{\mu\beta}^\alpha \Delta_{\nu\alpha}^\beta - \Delta_{\mu\nu}^\alpha \Delta_{\alpha\beta}^\beta).$$

One verifies that

$$(16) \quad \frac{\delta L_g}{\delta g_{\mu\nu}} = \frac{1}{8\pi} \chi G^{\mu\nu}.$$

Hence, if we take as the variational principle

$$(17) \quad \delta \int L(-\gamma)^{\frac{1}{2}} d\tau = 0 ,$$

where the $g_{\mu\nu}$ are given arbitrary (infinitesimal) variations vanishing on the boundary, we obtain the field equations (9).

It should be pointed out that the form of L is not uniquely determined. For example, one can add terms, the variations of which vanish as a consequence of Eq. (5). This provides one way of obtaining an energy-momentum density tensor for the gravitational field.

First we note that, if we verify the $\gamma_{\mu\nu}$ (the variations vanishing on the boundary) and write

$$(18) \quad \delta \int L(-\gamma)^{\frac{1}{2}} d\tau = \int \Theta^{\mu\nu} \delta \gamma_{\mu\nu} (-\gamma)^{\frac{1}{2}} d\tau ,$$

then $\Theta^{\mu\nu}$ satisfies the conservation equation

$$(19) \quad \Theta^{\mu\nu}{}_{,\nu} = 0 .$$

(This can be seen, *e.g.*, from the fact that, ignoring Eq. (5), we could add to L a term

$$\frac{1}{8\pi} P = \frac{1}{8\pi} \gamma^{\sigma\tau} P^{\alpha}{}_{\sigma\tau\alpha}$$

and then extend the variational principle (17) to include variations of $\gamma_{\mu\nu}$). One can interpret $\Theta^{\mu\nu}$ as the total energy-momentum density tensor, including that of the gravitational field. Let us assume that $L_m(-\gamma)^{\frac{1}{2}}$ does not depend on $\gamma_{\mu\nu}$. Making use of Eq. (15) for L_g , one finds that

$$(20) \quad 16\pi\Theta^{\mu\nu} = (\bar{g}^{\mu\nu} \gamma^{\sigma\tau} + \bar{g}^{\sigma\tau} \gamma^{\mu\nu} - \bar{g}^{\mu\sigma} \gamma^{\nu\tau} - \bar{g}^{\nu\sigma} \gamma^{\mu\tau})_{,\sigma\tau} .$$

This is essentially the expression obtained by Papapetrou [2]. If one uses it to calculate the gravitational energy-momentum density tensor, one finds that the latter contains second derivatives of $g_{\mu\nu}$. However, it appears desirable that derivatives no higher than the first should be present in this tensor for the following reason:

In order that g -space be flat, a sufficient condition is that

$$(21) \quad g_{\mu\nu,\sigma} = 0 ,$$

since then $\Delta^{\lambda}_{\mu\nu} = 0$ and, by (6), $R^{\lambda}_{\mu\nu\sigma} = 0$. However this condition is also

necessary, provided $g_{\mu\nu}$ is singularity-free and satisfies Eq. (11) and appropriate boundary conditions. For one can transform to Cartesian coordinates ($\gamma_{\mu\nu} = \eta_{\mu\nu}$), in which case Eq. (11) represents the De Donder or «harmonicity» condition in the usual form. Now Fock [6] has essentially shown that (with appropriate boundary conditions) the harmonicity condition permits only linear transformations of the metric. On the other hand, the metric $g_{\mu\nu} = \eta_{\mu\nu}$ is one for which $R^\lambda_{\mu\nu\sigma} = 0$, and Eq. (11) and (21) are satisfied. Taking into account the boundary conditions and the restriction to linear transformations, we see that, in Cartesian coordinates, the flat space g -metric satisfying (11) is unique and also satisfies (21). Since we are dealing with tensor equations, this result holds for any coordinates.

We see then that we can characterize the flatness or non-flatness of space, *i.e.*, the absence or presence of a real gravitational field (as distinguished from an inertial field) by the behavior of $g_{\mu\nu,\sigma}$. It seems therefore reasonable to insist that the energy and momentum of a gravitational field be expressible in terms $g_{\mu\nu}$ and $g_{\mu\nu,\sigma}$, without any higher derivatives.

One way of obtaining a gravitational energy-momentum density tensor involving only first derivatives is to get the «canonical» tensor from the Lagrangian density L_g of Eq. (15) by the standard procedure [7]. This gives our tensor counterpart to Einstein's pseudo-tensor. However, the latter has the well known difficulty of being asymmetrical and therefore unable to give a conservation law for the angular momentum of the field. We therefore look for a procedure that will give instead the tensor counterpart of the pseudo-tensor of Landau and Lifshitz [8].

Let us consider the variation

$$\delta \int L_g(-\lambda)^{\frac{1}{2}} d\tau,$$

where the $\gamma_{\mu\nu}$ are varied, and where

$$(22) \quad L_g = W^{\mu\nu\sigma\tau} P_{\mu\nu\sigma\tau},$$

with the understanding that, after the variation, we set $P_{\mu\nu\sigma\tau} = 0$. Here $W^{\mu\nu\sigma\tau}$ is an arbitrary tensor having the same symmetry properties as $P_{\mu\nu\sigma\tau}$. One finds that

$$(23) \quad \delta \int L_g(-\gamma)^{\frac{1}{2}} d\tau = 2 \int W^{\mu\nu\sigma\tau}_{;\rho\sigma} \delta \gamma_{\mu\nu} (-\gamma)^{\frac{1}{2}} d\tau,$$

so that if one writes

$$(24) \quad V^{\mu\nu} = 2 W^{\mu\nu\sigma\tau}_{;\rho\sigma},$$

we have

$$(25) \quad V^{\mu\nu}{}_{,\nu} = 0,$$

and $V^{\mu\nu}$ can be interpreted as the total energy-momentum density tensor. In particular, if we take

$$(26) \quad W^{\mu\nu\sigma} = \frac{1}{64\pi} (A^{\mu\nu} B^{\sigma\sigma} + A^{\sigma\sigma} B^{\mu\nu} - A^{\mu\sigma} B^{\nu\sigma} - A^{\nu\sigma} B^{\mu\sigma}),$$

where $A^{\mu\nu}$ and $B^{\mu\nu}$ are symmetrical tensors, we get

$$(27) \quad V^{\mu\nu} = \frac{1}{32\pi} (A^{\mu\nu} B^{\sigma\sigma} + A^{\sigma\sigma} B^{\mu\nu} - A^{\mu\sigma} B^{\nu\sigma} - A^{\nu\sigma} B^{\mu\sigma})_{,\sigma}.$$

To get $V^{\mu\nu} = \Theta^{\mu\nu}$, as given by (20), we can take $A^{\mu\nu} = \bar{g}^{\mu\nu}$, $B^{\mu\nu} = 2\gamma^{\mu\nu}$. On the other hand, to get the Landau-Lifshitz expression [8], we take $A^{\mu\nu} = B^{\mu\nu} = \bar{g}^{\mu\nu}$, so that

$$(28) \quad V^{\mu\nu} = \frac{1}{16\pi} (\bar{g}^{\mu\nu} \bar{g}^{\sigma\sigma} - \bar{g}^{\mu\sigma} \bar{g}^{\nu\sigma})_{,\sigma}.$$

We see then that, by a suitable choice of $W^{\mu\nu\sigma}$, we can have an L_γ which, when added to our previous L of Eq. (13), will enable us to obtain, by a variation of $\gamma_{\mu\nu}$, the total energy-momentum tensor (28). If, in analogy to the procedure of Landau and Lifshitz [8], we now write

$$(29) \quad V^{\mu\nu} = \chi^2 (T^{\mu\nu} + t^{\mu\nu}),$$

where $t^{\mu\nu}$ is the gravitational energy-momentum density tensor, and replace $T^{\mu\nu}$ by $-(1/8\pi)G^{\mu\nu}$, we get an explicit expression for $t^{\mu\nu}$. With eq. (11) holding, this is given by

$$(30) \quad 16\pi\chi^2 t^{\mu\nu} = \bar{g}^{\mu\alpha}{}_{,\beta} \bar{g}^{\nu\alpha}{}_{,\beta} + \frac{1}{2} \bar{g}^{\beta\alpha}{}_{,\mu} \bar{g}^{\alpha\beta}{}_{,\nu} - \bar{g}^{\mu\alpha}{}_{,\beta} \bar{g}^{\alpha\beta}{}_{,\nu} - \bar{g}^{\nu\alpha}{}_{,\beta} \bar{g}^{\alpha\beta}{}_{,\mu} - \chi_{,\mu} \chi_{,\nu} / \chi^2 + \bar{g}^{\mu\nu} L_g,$$

where a line under an index means raising or lowering it with $\bar{g}^{\mu\nu}$ or $\bar{g}_{\mu\nu}$ ($\bar{g}_{\mu\alpha} \bar{g}^{\nu\alpha} = \delta_\mu^\nu$) after the differentiation, and L_g is now given by

$$(31) \quad L_g = \frac{1}{2} g^{\alpha\beta}{}_{,\gamma} g^{\gamma\gamma}{}_{,\beta} - \frac{1}{2} g^{\alpha\beta}{}_{,\gamma} g^{\alpha\beta}{}_{,\gamma} + \frac{1}{2} \chi_{,\alpha} \chi_{,\alpha} / \chi^2.$$

We see that (30), being a tensor and containing only covariant γ -derivatives, describes the energy and momentum of the real gravitational field, not those of the inertial field. The inertial field makes itself felt in the conservation law, eq. (25). This is a true conservation equation if

$\gamma_{\mu\nu} = \eta_{\mu\nu}$. Otherwise the left-hand member is in general not an ordinary divergence and its integral over a (4-dimensional) volume cannot be converted into a surface integral; a term is present involving $\Gamma_{\mu\nu}^\lambda$, which describes the effect of the inertial forces.

It appears that we now have a complete and consistent formalism for describing the gravitational field. If we look back at the field equations we have chosen, *i.e.*, those of Einstein, we see that they have a number of advantages:

1) The existence of the identities (10) among them leads to corresponding equation for the energy-momentum density tensor $T_{\mu\nu}$ and thus provides explicit equations of motion for matter.

2) The identities enable us to impose the conditions (11) in order to relate the $g_{\mu\nu}$ to the $\gamma_{\mu\nu}$. It then follows that we characterize flatness of space by Eq. (21), which contains only first derivatives, instead of having to make use of the Riemann-Christoffel tensor containing second derivatives.

3) It is accordingly possible to find a symmetrical energy-momentum density tensor for the gravitational field involving only first derivatives of the g -metric.

4) The field equations can be written (as was done by Einstein) in a form independent of $\gamma_{\mu\nu}$. This can be interpreted as indicating the existence of an invariance principle [3], according to which the field equations are invariant under transformations of $\gamma_{\mu\nu}$, regarded as « gauge transformations ». Eq. (11), which depends on $\gamma_{\mu\nu}$, can be considered as a convenient convention, other conventions being possible.

The field equations of general relativity determine the motion of material bodies present in the gravitational field. In particular, the motion of a test particle is given by the equation of the geodesic. Using the present formalism, this can be written

$$(32) \quad \frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} + \Delta_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0.$$

In the expression on the left, the third term can be considered as describing the effect of the real gravitational field, while the second term can be thought of as associated with the inertial field. The similarity of these two terms, in consequence of which they may cancel each other in certain circumstances, or be indistinguishable in others, is a manifestation of the equivalence principle of general relativity.

If one multiplies (32) by m_0 , the mass constant characterizing the test particle, and one takes σ instead of s as the independent variable,

one can write [1] the equation in the form

$$(33) \quad \frac{d}{d\sigma} \left(m \frac{dx^\mu}{d\sigma} \right) + m \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} = -m \Delta_{\alpha\beta}^\mu \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma},$$

where

$$(34) \quad m = m_0 \frac{d\sigma}{ds} = m_0 / \left(g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \right)^{\frac{1}{2}}.$$

Eq. (33) is particularly appropriate for the viewpoint that physical space-time is described by the γ -space.

3. - Gravitational waves.

In conclusion, it seems desirable to add a few brief remarks about gravitational waves.

If, in the usual form of general relativity, one has a situation in which space-time is flat at infinity, and one takes coordinates which satisfy the De Donder condition and are Galilean at infinity, the formalism is identical with that of bi-metric relativity where one has chosen coordinates so that $\gamma_{\mu\nu} = \eta_{\mu\nu}$. A great deal of work has been done in general relativity for the conditions mentioned, and this can be carried over into bi-metric relativity.

Reference might be made to the case of a weak field ($g_{\mu\nu}$ nearly equal to $\gamma_{\mu\nu}$). Landau and Lifshitz [8] have shown that the energy density of a plane gravitational wave, as calculated essentially from Eq. (30), is positive. This is as it should be, if one thinks of gravitational radiation as analogous to electromagnetic radiation, in that it carries away energy from the radiating system. However, in the general case, the gravitational energy density, as given by (30), will not necessarily be positive. Here the analogy with electromagnetism ends. One can convince oneself that, for two masses of the same sign to attract each other, the energy density cannot be expected to be positive everywhere.

In recent years considerable work has been done on the quantization of the gravitational field. It is not known whether the gravitational fields in nature are quantized. If it should turn out that one must quantize gravitation, so that $g_{\mu\nu}$ becomes a quantum-mechanical operator, there might arise some conceptual difficulties in the usual form of general relativity: there is the danger that, with the geometry quantized, it might not be possible to order events in a spatio-temporal framework or to give meaning to the space-time coincidences of events. In that case there might

be some advantage to bi-metric relativity, since it can provide an « unquantized » flat-space framework in which the quantized fields are situated and physical events take place.

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INTERVENTI E DISCUSSIONI

— P. G. BERGMANN:

1) Introduction of a flat metric is not equivalent to the construction of tetrads, but of « holonomic » tetrads.

2) If the flat metric is taken seriously, then the principle of general covariance is destroyed. If not, then there exists an infinity of mappings $R \rightleftharpoons E$.

— N. ROSEN:

1) One can introduce a flat metric without constructing tetrads. On the other hand, if one begins with tetrads, one obtains a local flat metric, which can be taken to be $\eta_{\mu\nu}$, as Möller does, or can be generalized to $\gamma_{\mu\nu}$. The co-ordinate system used is the same whether one is dealing with $g_{\mu\nu}$ or $\gamma_{\mu\nu}$ and the same co-ordinate differentials occur in the ds and $d\sigma$. It is not the tetrads which are « holonomic », but the dx^μ , since these are differentials of functions of space-time.

2) The principle of covariance is maintained, since arbitrary co-ordinate transformations are allowed. It is to be expected that the proposed imposition of four covariant auxiliary conditions (plus suitable boundary conditions) should give a unique relation between $g_{\mu\nu}$ and $\gamma_{\mu\nu}$. This could be integrated as a unique mapping $R \rightleftharpoons E$.

— A. KOMAR:

The introduction of a second metric is essentially equivalent to the introduction of a preferred vierbein frame of a rather special sort: namely, the four vierbein fields introduced are hypersurface-orthogonal. The auxiliary equations introduced in order to further specify the second metric can be stated equally

well as conditions on the vierbein field. The approach one prefers is purely a matter of taste.

Having thus introduced a preferred frame two possibilities arise: 1) either the choice of the preferred frame is purely a convenience and has no physical significance, or 2) the preferred frame is really *physically* preferred. In the case 1) the physical conclusions should not depend on the choice of the frame, so that we should have an effective « gauge group » of transformations available which leave invariant all observable expressions. In this event it would be preferable to carry out all derivations in an explicitly « gauge-invariant » fashion, that is, not to introduce the preferred frame to begin with. In case 2) the existence of a physically preferred flat frame destroyed the general covariance of the theory, for it introduces preferred status for the Lorentz group.

Of course we are still free to employ curvilinear co-ordinates in flat space (*e.g.* we can and do employ polar co-ordinates in Maxwell's equations) but the preferred status of the Lorentz group can always be recovered in an invariant fashion due to the existence of ten independent Killing vector fields having the appropriate commutation relations. Thus in case 2) we are no longer dealing with the Einstein theory, but rather with some cumbersome, nonlinear Lorentz-covariant theory having many more observables due to the additional geometric structure introduced *ab initio*.

— N. ROSEN:

If the two metrics are given one does not have to introduce tetrads. However, if one does introduce them, it appears that they are not uniquely determined, since one has only ten conditions for the sixteen components of the tetrad vectors. As for the question of covariance, the formulation is clearly covariant, and situations can arise when it is convenient to use co-ordinates other than Galileian.

— P. HAYAS:

There exists an interesting analogue to Dr. Rosen's procedure: the use of two affine connections arises naturally if one wishes to give a generally covariant formulation of Newton's theory of gravitation. Such a formulation can be achieved by introducing in addition to an affine connection appropriate for flat space another connexion whose curvature tensor does not necessarily vanish and which satisfies field equations of the same form as Einstein's. This method is due to Friedrichs (*Math. Ann.*, **98**, 566 (1928) and has recently been extended and compared with Rosen's approach (to appear in *Rev. Mod. Phys.* (October 1964)).

— D. IVANENKO:

If I may ask some further clarification of the statement in your interesting report that the quadruples (tetrads) do not appear necessary in the pure gravitational theory. In my opinion the mere fact that termionic spinor fields do interact via tetrads and not Riemann-Einsteinian metrical components prove that one cannot avoid introducing of tetrads as fundamental entities in gravidynamics.

— C. CATTANEO:

Je voudrais signaler qu'en introduisant, à côté de la métrique riemannienne une deuxième métrique minkowskienne, M. Bonazzola a pu obtenir un champ de 4 vecteurs à divergence covariante nulle (divergence calculée par rapport à la *métrique riemannienne*). Ce champ donne lieu à des intéressantes interprétations physiques.

[Alla discussione parteciparono anche C. MØLLER, I. ROBINSON, M. A. TONNELAT.]

Sur le formalisme hamiltonien en relativité générale.

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1. - Introduction et résumé.

On sait que la théorie des champs possède deux caractéristiques essentielles: ses équations dérivent de principes variationnels, et elles sont invariantes pour certain groupes ou pseudo-groupes de transformations. Ceux-ci limitent le choix des lagrangiens et peuvent, comme en Relativité générale, entraîner des contraintes. Des composantes du champ sont alors arbitraires et il est utile de les mettre en évidence aussi tôt que possible. C'est ce que nous voulons montrer ci-dessous.

La séparation des composantes y^A du champ en les variables arbitraires e^a et dynamiques q^K est déterminée par la forme des transformations infinitésimales (t.i.) des y . Une conséquence des identités de Bianchi permet de construire un lagrangien indépendant des dérivées des e par rapport au temps. Une autre introduit les fonctions $\mathcal{H}_a$ des variables canoniquement conjuguées (q, p) , dont les t.i. sont nécessairement canoniques, de génératrices $\mathcal{H}_a$, afin d'assurer l'invariance des équations hamiltoniennes. Les parenthèses de Poisson des $\mathcal{H}$ sont des fonctions linéaires de ces génératrices, $\mathcal{H}_a G_{bc}^a(q, p)$; les G se tirent des t.i. des e .

A la fin de l'exposé, la méthode est appliquée brièvement au cas de la Relativité générale.

2. - Hypothèses.

Soient

$$(1) \quad y^A = y^A(x) \quad A = 1, \dots, N$$

les composantes du champ au point de coordonnées

$$(x) = (x^0, x^1, \dots, x^m)$$

d'une variété réelle continue à $(m+1)$ dimensions.

Par hypothèse, les équations du champ peuvent se mettre sous la forme lagrangienne

$$(2) \quad \mathcal{L}_A \equiv \frac{\partial \mathcal{L}}{\partial y^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial y_\mu^A} = 0 \quad \mu = 0, 1, \dots, m$$

où $\mathcal{L}$ est une fonction des y^A et de leurs dérivées partielles y_μ^A par rapport aux x^μ .

De plus, ces équations sont invariantes pour un groupe ou pseudo-groupe de transformations. Les composantes du champ sont les variables d'une de ses réalisations. Toutes celles utilisées jusqu'à présent subissent des transformations infinitésimales (t.i.) du type

$$(3) \quad y'^A(x) - y^A(x) = \delta y^A = \chi_a^A \xi^a + \varphi_a^{A\mu} \xi_\mu^a + \theta_a^{A\mu\nu} \xi_{\mu\nu}^a \\ a = 1, \dots, n; \quad \mu, \nu = 0, 1, \dots, m.$$

Les ξ sont des formes linéaires

$$(4) \quad \xi^a = U_b^a(u, u_\mu) \delta u^b \quad a, b = 1, \dots, n$$

dont les coefficients U sont des fonctions des n paramètres (canoniques) $u^c(x)$ et de leurs dérivées premières u_μ^c par rapport aux x^μ .

$$\xi_\mu^a = \partial_\mu \xi^a = \partial_\mu U_b^a \delta u^b + U_b^a \delta u_\mu^b, \\ \xi_{\mu\nu}^a = \partial_\mu \partial_\nu \xi^a.$$

Les θ sont des constantes, les φ des fonctions des y , les χ des fonctions des y et de leurs dérivées premières y_μ par rapport aux x^μ . Des termes en $\xi_{\mu\nu}^a$ existent en Relativité générale lorsque les accolades de Christoffel sont prises comme composantes du champ; ils élargissent donc considérablement le domaine des applications. Cependant, nous les supprimerons ici pour simplifier l'exposé; d'où

$$(5) \quad \delta y^A = \chi_a^A \xi^a + \varphi_a^{A\mu} \xi_\mu^a.$$

3. - Identités.

En vertu de la seconde hypothèse, on pose que la variation de $\mathcal{L}$, due à (5), est une divergence;

$$(6) \quad \delta \mathcal{L} = \partial_\lambda \mathcal{K}^\lambda$$

$$(7) \quad \mathcal{K}^\lambda = \mathcal{K}_a^\lambda \xi^a + \mathcal{K}_a^{\lambda\mu} \xi_\mu^a + \mathcal{K}_a^{\lambda\mu\nu} \xi_{\mu\nu}^a$$

quels que soient les ξ et leurs dérivées. On obtient ainsi les identités en les y et leurs dérivées [1]

$$\begin{aligned}
 (8) \quad & \mathcal{L}_A \chi_a^A = \partial_\lambda f_a^\lambda \\
 (9) \quad & \mathcal{L}_A \varphi_a^{A\lambda} - f_a^\lambda = \partial_\mu r_a^{\mu\lambda} \\
 (10) \quad & r_a^{\lambda\mu} + r_a^{\mu\lambda} = 2 \partial_\nu \mathcal{K}_a^{\nu\lambda\mu} \\
 (11) \quad & S \mathcal{K}_a^{\lambda\mu\nu} \equiv \mathcal{K}_a^{\lambda\mu\nu} + \mathcal{K}_a^{\mu\nu\lambda} + \mathcal{K}_a^{\nu\lambda\mu} = 0
 \end{aligned}$$

où $\mathcal{L}_A^\mu$ est la dérivée de $\mathcal{L}$ par rapport à y_μ^A et

$$\begin{aligned}
 (12) \quad & f_a^\lambda = \mathcal{K}_a^\lambda - \mathcal{L}_A^\lambda \chi_a^A \\
 (13) \quad & r_a^{\mu\lambda} = \mathcal{K}_a^{\mu\lambda} - \mathcal{L}_A^\mu \varphi_a^{A\lambda} .
 \end{aligned}$$

Dans (9), r peut être remplacé par la grandeur antisymétrique [1]

$$(14) \quad \chi_a^{\mu\lambda} = \frac{1}{2}(r_a^{\mu\lambda} - r_a^{\lambda\mu}) + \frac{1}{3} \partial_\nu (\mathcal{K}_a^{\mu\lambda\nu} - \mathcal{K}_a^{\lambda\mu\nu}) .$$

De (9) et (14), on tire

$$(15) \quad \partial_\lambda f_a^\lambda = \partial_\lambda (\mathcal{L}_A \varphi_a^{A\lambda}) ,$$

ce qui permet de remplacer (8) par

$$(8') \quad \mathcal{L}_A \chi_a^A = \partial_\lambda (\mathcal{L}_A \varphi_a^{A\lambda}) .$$

Les dérivées troisièmes des composantes du champ y figurent linéairement. Leurs coefficients doivent donc être nuls:

$$(16) \quad S \frac{\partial^2 \mathcal{L}}{\partial y_\lambda^A \partial y_\mu^B} \varphi_a^{B\mu} = 0 ,$$

où S désigne une sommation sur $\lambda\mu\nu$ comme en (11). Remarquons que les $\mathcal{K}$ peuvent aussi être éliminés de (6):

$$(17) \quad \delta \mathcal{L} = \partial_\lambda (\mathcal{L}_A^\lambda \delta y^A + \mathcal{L}_A \varphi_a^{A\lambda} \xi^a) .$$

4. - Variables adaptées.

Le formalisme hamiltonien suppose l'existence, dans la variété à $(m+1)$ dimensions, d'une famille d'hypersurfaces à m dimensions telles qu'il en passe une et une seule par chaque point. Nous poserons pour

simplifier que ce sont les hypersurfaces d'équations

$$t \equiv x^0 = \text{constante}.$$

Les dérivées par rapport à t seront indiquées par un point;

$$y_0^A \equiv \dot{y}^A, \quad \xi_0^a \equiv \dot{\xi}^a,$$

et nous écrirons, au lieu de (5),

$$(18) \quad \delta y^A = \chi_a^A \xi^a + \varphi_a^{Ar} \xi_r^a + \varphi_a^A \dot{\xi}^a. \quad (r = 1, \dots, m).$$

En vertu de (16), les n combinaisons linéaires

$$(19) \quad p_A \varphi_a^A \equiv \pi_a$$

des moments

$$(20) \quad p^A = \frac{\partial \mathcal{L}}{\partial \dot{y}^A}$$

conjugués aux y ne dépendent pas des $\dot{y}$.

M. KATZ ([2] p. 68) a montré que ces contraintes sont éliminées automatiquement lorsqu'on exploite les conditions d'intégrabilité des t.i. du groupe d'invariance.

Ainsi, les conditions d'intégrabilité de

$$(21) \quad \delta' y^A = \varphi_a^A \dot{\xi}^a \quad (\xi^a = 0, \xi_r^a = 0)$$

assurent l'existence de $(N-n)$ solutions distinctes $q^K = q^K(y)$ des équations

$$(22) \quad \frac{\partial q^K}{\partial y^A} \varphi_a^A = 0$$

et de n solutions distinctes $e^a = e^a(y)$ des équations

$$(23) \quad \frac{\partial e^a}{\partial y^A} \varphi_b^A = D_b^a(e)$$

où D est une matrice régulière telle que

$$(24) \quad \delta' e = D_b^a \dot{\xi}^b$$

définit une réalisation du même sous-groupe que celui considéré en (21).

Dans ces nouvelles variables q, e , les t.i. (18) deviennent

$$(25) \quad \delta q^K = \chi_a^K \xi^a + \varphi_a^{Kr} \xi_r^a$$

$$(26) \quad \delta e^a = \chi_b^a \xi^b + \varphi_b^{Ar} \xi_r^b + D_b^a \dot{\xi}^b$$

où

$$(27) \quad \chi_a^K = \frac{\partial q^K}{\partial y^A} \chi_a^A, \quad \varphi_a^{Kr} = \frac{\partial q^K}{\partial y^A} \varphi_a^{Ar},$$

$$(28) \quad \chi_b^a = \frac{\partial e^a}{\partial y^A} \chi_b^A, \quad \varphi_b^{Ar} = \frac{\partial e^a}{\partial y^A} \varphi_b^{Ar}.$$

Il est facile de voir que les conditions d'intégrabilité de (5) et (26) imposent à

$$\chi_a^K \quad \text{et} \quad \chi_b^a - \dot{D}_b^a$$

d'être indépendantes des e .

Les moments conjugués aux e, q ,

$$(29) \quad p_a = \frac{\partial \mathcal{L}}{\partial \dot{e}^a}, \quad p^K = \frac{\partial \mathcal{L}}{\partial \dot{q}^K}$$

ne dépendent pas des e . Cela résulte des relations (19) qui se réduisent maintenant à

$$(30) \quad p_b D_a^b = \pi_a(e, q, e_r, q_r).$$

Les identités (8') deviennent

$$(31) \quad \mathcal{L}_b \chi_a^b + \mathcal{L}_K \chi_a^K = \partial_r (\mathcal{L}_b \varphi_a^{br} + \mathcal{L}_K \varphi_a^{Kr}) + (\mathcal{L}_b D_b^a)^{\bullet}$$

où

$$(32) \quad \mathcal{L}_a = L_a - \dot{p}_a, \quad \mathcal{L}_K = L_K - \dot{p}_K,$$

$$(33) \quad L_a = \frac{\partial \mathcal{L}}{\partial e^a} - \partial_r \frac{\partial \mathcal{L}}{\partial e_r^a}, \quad L_K = \frac{\partial \mathcal{L}}{\partial q^K} - \partial_r \frac{\partial \mathcal{L}}{\partial q_r^K}.$$

A cause de (16), les dérivées troisièmes du champ ne figurent pas dans (31). Les dérivées secondes $\ddot{e}$ disparaissent sous les conditions

$$(34) \quad \frac{\partial p_a}{\partial e^b} - \partial_r \frac{\partial p_a}{\partial e_r^b} = \frac{\partial p_b}{\partial e^a}$$

dont la solution générale

$$(35) \quad p_a = \frac{\partial l}{\partial e^a} - \partial_r \frac{\partial l}{\partial e^a_r}$$

peut aussi s'écrire

$$(36) \quad \frac{\partial}{\partial \dot{e}^a} (\partial_\alpha l^\alpha)$$

avec $l^0 = l$ et

$$(37) \quad \frac{\partial l_\alpha}{\partial e^c_\beta} + \frac{\partial l_\beta}{\partial e^c_\alpha} = 0, \quad \frac{\partial l^\alpha}{\partial q^\kappa_\beta} + \frac{\partial l^\beta}{\partial q^\kappa_\alpha} = 0.$$

Grâce à la fonction lagrangienne

$$(38) \quad \bar{\mathcal{L}} = \mathcal{L} - \partial_\alpha l^\alpha,$$

qui ne contient pas les $\dot{e}$, les équations lagrangiennes sont réparties en deux systèmes

$$(39) \quad \bar{L}_a \equiv \frac{\partial \bar{\mathcal{L}}}{\partial e^a} - \partial_r \frac{\partial \bar{\mathcal{L}}}{\partial e^a_r} = 0$$

$$(40) \quad \mathcal{L}_\kappa \equiv \frac{\partial \bar{\mathcal{L}}}{\partial q^\kappa} - \partial_\mu \frac{\partial \bar{\mathcal{L}}}{\partial q^\kappa_\mu} = 0.$$

Le second est celui des équations dynamiques. Le premier rassemble les équations de conditions, que l'on peut aussi écrire à l'aide de la fonction lagrangienne initiale puisque

$$(41) \quad \bar{L}_a = \mathcal{L}_A \frac{\partial y^A}{\partial e^a} \quad \text{ou} \quad \mathcal{L}_A \varphi^A_a = \bar{L}_b D^b_a.$$

Les $e^a(x)$ sont les n fonctions arbitraires du problème.

5. - Formalisme hamiltonien.

A cause du rôle privilégié joué par une des coordonnées, le temps, une bonne partie de ce formalisme est commune à la Mécanique classique et à la physique des champs. On peut même pousser l'analogie très loin, en considérant les coordonnées spatiales comme des indices, contenus implicitement dans les indices $A, B, \dots, a, b, \dots$ qui ne prenaient jusqu'ici qu'un nombre fini de valeurs, et en impliquant dans toute sommation

une intégration sur ces coordonnées [2]. C'est pourquoi, dans ce paragraphe, l'exposé sera développé en termes de la Mécanique classique.

Pour simplifier les notations, le lagrangien non surligné sera déjà supposé indépendant des $\dot{e}$. Les équations

$$(42) \quad p_K = \frac{\partial \mathcal{L}}{\partial \dot{q}^K}$$

sont, par hypothèse, résolubles par rapport aux $\dot{q}$:

$$(43) \quad \{\dot{q}^K\} = \dot{q}^K(e, q, p) .$$

Dans l'identité (31), c'est-à-dire ici

$$(44) \quad L_b \chi_a^b + \mathcal{L}_K \chi_a^K = (L_b D_a^b)^*,$$

les $\ddot{q}$ et e figurent linéairement. Leurs coefficients sont donc nécessairement nuls. Cela donne ([2], p. 73), pour $\ddot{q}$,

$$(45) \quad \frac{\partial p^K}{\partial e^a} + \frac{\partial p^K}{\partial \dot{q}^L} \bar{\chi}_a^L = 0$$

avec

$$(46) \quad \bar{\chi}_a^L = \chi_b^L C_a^b ,$$

où C est la matrice inverse de D , et, pour e , la même identité, avec L_a au lieu de p_K . De (45),

$$(47) \quad \bar{\chi}_a^L = \frac{\partial}{\partial e^a} \{\dot{q}^L\}$$

et, avec L_a ,

$$(48) \quad \{L_a\} \equiv -\mathcal{H}_a(q, p)$$

est une fonction des q, p , non des e .

De (45) encore,

$$(49) \quad \bar{\chi}_a^K = \frac{\partial \mathcal{H}_a}{\partial p_K} = \bar{\chi}_a^K(q, p) .$$

Les t.i. (25) et (26) se réduisent à

$$(50) \quad \delta q^K = \chi_a^K \xi^a$$

$$(51) \quad \delta e^a = \chi_b^a \xi^b + D_b^a \dot{\xi}^b , \quad (51)$$

ou encore, par le changement ([2], p. 69)

$$(52) \quad \xi^a = C_b^a \bar{\xi}^b, \quad \bar{\xi}^a = D_b^a \xi^b,$$

$$(53) \quad \delta q^K = \bar{\chi}_a^K \bar{\xi}^a$$

$$(54) \quad \delta e^a = \bar{\chi}_b^a \bar{\xi}^b + \dot{\bar{\xi}}^a$$

où

$$(55) \quad \bar{\chi}_b^a = (\chi_c^a - \dot{D}_c^a) C_b^c.$$

Rappelons que ces $\bar{\chi}$ ne dépendent pas des $\dot{e}$.

En vertu de (45), la t.i. des p

$$(56) \quad \delta p_K = \bar{\chi}_{Ka} \bar{\xi}^a$$

ne contient pas de termes en $\dot{\bar{\xi}}$.

La propriété (17) devient ici

$$(57) \quad \delta L = (p_K \delta q^K - \mathcal{H}_a \bar{\xi}^a)^* ;$$

d'où, pour la fonction hamiltonienne

$$(58) \quad H \equiv p_K \{ \dot{q}^K \} - \{ L \}$$

$$(59) \quad \delta H = \left(\bar{\chi}_{Ka} + \frac{\partial \mathcal{H}_a}{\partial q^K} \right) \dot{q}^K \bar{\xi}^a + \mathcal{H}_a \dot{\bar{\xi}}^a.$$

D'autre part, les deux expressions générales

$$(60) \quad \delta H = \frac{\partial H}{\partial e^a} \delta e^a + \frac{\partial H}{\partial q^K} \delta q^K + \frac{\partial H}{\partial p_K} \delta p_K$$

$$(61) \quad \delta H = - \frac{\partial L}{\partial e^a} \delta e^a - \frac{\partial L}{\partial q^K} \delta q^K + \dot{q}^K \delta p_K$$

donnent les relations habituelles

$$(62) \quad \frac{\partial H}{\partial e^a} = - \frac{\partial L}{\partial e^a}, \quad \frac{\partial H}{\partial q^K} = - \frac{\partial L}{\partial q^K}, \quad \frac{\partial H}{\partial p_K} = \dot{q}^K.$$

Les deux t.i. (59) et (60) sont identiques quels que soient les $\zeta, \dot{\zeta}$ sous la condition

$$(63) \quad (\mathcal{H}_a, H) = \mathcal{H}_b \bar{\chi}_a^b.$$

où a été introduit la notation des parenthèses de Poisson

$$(64) \quad (A, B) = \frac{\partial A}{\partial q^\kappa} \frac{\partial B}{\partial p_\kappa} - \frac{\partial B}{\partial q^\kappa} \frac{\partial A}{\partial p_\kappa}.$$

De la première relation (62) et de (48) on déduit que

$$(65) \quad H = \mathcal{H}_a e^a + h(q, p)$$

d'où (63),

$$(66) \quad (\mathcal{H}_a, \mathcal{H}_b) e^b + (\mathcal{H}_a, h) = \mathcal{H}_b \bar{\chi}_a^b.$$

On en tire que $\bar{\chi}_b^a$ est de la forme

$$(67) \quad \bar{\chi}_b^a = G_{bc}^a(q, p) \cdot e^c + k_b^a(q, p)$$

et

$$(68) \quad (\mathcal{H}_b, \mathcal{H}_c) = \mathcal{H}_a G_{bc}^a$$

$$(69) \quad (\mathcal{H}_b, h) = \mathcal{H}_a k_b^a,$$

d'où aussi

$$(70) \quad \delta \mathcal{H}_a = \mathcal{H}_c G_{ab}^c \bar{\xi}^b + \bar{\chi}_a^k \left(\bar{\chi}_{kb} + \frac{\partial \mathcal{H}_b}{\partial q^\kappa} \right) \bar{\xi}^b.$$

Les équations du mouvement se répartissent, comme en (39), (40), en équations de conditions

$$(71) \quad \mathcal{H}_a = 0$$

et en équations dynamiques dont la forme hamiltonienne est

$$(72) \quad \dot{q}^\kappa = \frac{\partial H}{\partial p_\kappa}, \quad \dot{p}_\kappa = - \frac{\partial H}{\partial q^\kappa},$$

avec (65).

Exprimons que ces équations sont invariantes pour les t.i. (53), (54), (56). Tout d'abord pour les variations δ' , ($\bar{\xi} = 0$, $\dot{\bar{\xi}} \neq 0$)

$$(73) \quad \delta' \left(\dot{p}_\kappa + \frac{\partial H}{\partial q^\kappa} \right) = \left(\chi_{\kappa a} + \frac{\partial \mathcal{H}_a}{\partial q^\kappa} \right) \dot{\bar{\xi}}^a.$$

Il faut donc que

$$(74) \quad \bar{\chi}_{\kappa a} = - \frac{\partial \mathcal{H}_a}{\partial q^\kappa}$$

tout au moins en vertu des équations (71), (72). De (74) et (71) résulte l'invariance de (71) pour les t.i. complètes, et par un calcul simple on vérifie également l'invariance de (72) pour toutes ces t.i.

Les variations de q, p , sont donc des t.i. de contact de générateur (cfr. [3] et [4])

$$(75) \quad J = \mathcal{H}_a \bar{\xi}^a$$

$$(76) \quad \delta q^K = (q^K, J), \quad \delta p_K = (p_K, J).$$

De (69) on tire aussi que h est un invariant du groupe; on peut d'ailleurs toujours lier les fonctions arbitraires e aux paramètres $u, \bar{u}$ du groupe de telle sorte que $h^a_a = 0$.

6. - Cas de la relativité générale.

La variété est riemannienne à quatre dimensions, R_4 , localement minkowskienne (signature $(+2)$).

Les g^A sont ici les dix composantes $g_{\mu\nu}$ du tenseur métrique. Par les changements infinitésimaux de coordonnées

$$(77) \quad x'^\mu = x^\mu + \xi^\mu(x)$$

les $g_{\mu\nu}$ subissent les t.i.

$$(78) \quad g'_{\mu\nu}(x) - g_{\mu\nu}(x) = \delta g_{\mu\nu} = -g_{\mu\nu\alpha} \xi^\alpha - g_{\mu\alpha} \xi^\alpha_{,\nu} - g_{\nu\alpha} \xi^\alpha_{,\mu}.$$

(Un indice grec ajouté à un symbole défini précédemment signifie une dérivée partielle par rapport à une coordonnée).

On voit directement que les six fonctions

$$(79) \quad q_{rs} = g_{rs} \quad r, s = 1, 2, 3$$

satisfont aux équations (22). Il est naturel d'utiliser les notations de l'espace riemannien R_3 , localement euclidien, de tenseur métrique (79). La dérivée covariante sera notée D_r et les indices latins $r, s, \dots$, seront élevés et abaissés de la manière habituelle:

$$\xi_r = q_{rs} \xi^s, \quad \xi^r = q^{rs} \xi_s, \dots$$

Avec ces notations ([2], p. 86),

$$(80) \quad \delta q_{rs} = -D_r(\xi_s + g_{0s}\xi^0) - D_s(\xi_r + g_{0r}\xi^0) + 2\Omega_{rs}\xi^0/\sqrt{-g^{00}}$$

où

$$(81) \quad \Omega_{rs} = \sqrt{-g^{00}}(D_r g_{0s} + D_s g_{0r} - \dot{q}_{rs})/2 = -I_{rs}^0/\sqrt{-g^{00}}$$

est le second tenseur fondamental de R_3 . (Les I sont les symboles de Christoffel dans R_4).

Les variables arbitraires se tirent immédiatement de

$$\begin{aligned} \delta g_{0r} &= -(\xi_r + g_{0r} \xi^0) - g_{0rs} \xi^s - g_{0s} \xi_r^s - g_{00} \xi_r^0 + \dot{q}_{rs} \xi^s \\ \delta g^{00} &= -\dot{g}^{00} \xi^0 - g_r^{00} \xi^r + 2g^{0r} \xi_r^0 + 2g^{00} \xi^0. \end{aligned}$$

On a

$$(82) \quad e^0 = (-g^{00})^{-\frac{1}{2}}, \quad e_r = g_{0r}, \quad e^r = -g^{0r}/g^{00}$$

et

$$(83) \quad \bar{\xi}^0 = -e^0 \xi^0, \quad \bar{\xi}^r = -\xi^r - e^r \xi^0$$

$$(84) \quad D_\alpha^0 = -e^0 \delta_\alpha^0, \quad D_\alpha^r = -\delta_\alpha^r - e^r \delta_\alpha^0.$$

Désignons par $\mathcal{L}_{\mu\nu}$ la dérivée variationnelle de la fonction lagrangienne par rapport à $g^{\mu\nu}$. De (41), (48) et (84)

$$(85) \quad 2\mathcal{L}_{0r} g^{0r} \equiv 2\mathcal{L}_0^0 = -e^0 \bar{L}_0 - e^r \bar{L}_r = e^0 \mathcal{H}_0 + e^r \mathcal{H}_r$$

$$(86) \quad 2\mathcal{L}_{rv} g^{0v} \equiv 2\mathcal{L}_r^0 = -\bar{L}_r = \mathcal{H}_r.$$

La seconde forme fondamentale joue un rôle important dans cette étude ([2], p. 22) et [5]. Rappelons que

$$\begin{aligned} \mathcal{H}_0 &= \sqrt{-q}(\Omega_r^s \Omega_s^r - \Omega_r^r \Omega_s^s + {}^3R) \\ \mathcal{H}_r &= 2\sqrt{-q}(D_r \Omega_s^s - D_s \Omega_r^s), \end{aligned}$$

où 3R est l'invariant de courbure de R_3 .

On en tire la valeur des moments conjugués aux q_{rs} ,

$$p^{rs} = -\sqrt{-q}(\Omega^{rs} - q^{rs}\Omega_t^t);$$

d'où les $\mathcal{H}$ en fonctions des (q, p) et de leurs dérivées spatiales.

Des identités du Sect. 3, seules (8') ont été utilisées jusqu'ici. Mais la fonction lagrangienne est, dans cette application, une densité pour les changements linéaires homogènes des coordonnées. Une identité (9) en résulte, avec

$$(87) \quad \mathcal{H}_\beta^\lambda = -\mathcal{L}\delta_\beta^\lambda \quad \text{et} \quad \mathcal{H}_\beta^{\mu\lambda} = 0,$$

Pour $\lambda=0$ le second membre de (9) est une divergence spatiale, et cette identité fournit le lien entre la fonction hamiltonienne canonique f_0^0 et l'expression (85) qui est la fonction hamiltonienne obtenue par Dirac [6].

L'emploi de fonctions lagrangiennes qui sont des densités pour des changements plus larges de coordonnées permet d'utiliser également le autres identités du Sect. 3.

R É F É R E N C E S

- [1] Voir notamment, J. GÉHÉNTAU: *Colloque sur les Théories Relativistes de la Gravitation* (Royaumont, 1959); et les références qui s'y trouvent.
- [2] J. KATZ: *Thèse de Doctorat*, Faculté des Sciences de l'Université de Bruxelles (1964).
- [3] J. L. ANDERSON and P. G. BERGMANN: *Phys. Rev.*, **83**, 1018 (1951).
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- [5] J. L. ANDERSON: Stevens Institute of Technology, Hoboken, New Jersey, SIT-P96 (4/63).
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INTERVENTI E DISCUSSIONI

— P. G. BERGMANN:

In general relativity in the usual formalism, the « *c*-numbers » ξ^a describe a subgroup of the larger group in which the ξ^a may also depend on the field variables. But the commutation of two different *c*-number ξ -fields is not completely defined on a hypersurface $x^0 = \text{const}$. By contrast, the commutator between two sets of Dirac (ξ^r, ξ^z) is defined on a hypersurface (knowledge of $\dot{\xi}$ is not required), but the group property of *c*-numbers (ξ^r, ξ^z) is lost.

— J. GÉHÉNTAU:

Thank you for your remark which, I think, is a very general one. I would like to add one remark related to the last sentence of my report. Let us note $h_{(\alpha)\mu}(x)$ the covariant components of a tetrad field,

$$h_{(\alpha)[\mu\nu]} \equiv h_{(\alpha)\mu,\nu} - h_{(\alpha)\nu,\mu}$$

(, μ = partial derivative with respect to x^μ),

$$h_{(\alpha\beta\gamma)} \equiv h_{(\alpha)[\mu\nu]} h_{(\beta)}^\mu h_{(\gamma)}^\nu.$$

Due to (77)

$$\delta h_{(\alpha)\mu} = - h_{(\alpha)\mu,\nu} \xi^\nu - h_{\alpha(\nu} \xi_{\mu)}^\nu$$

$$\delta h_{(\alpha\beta\gamma)} = - h_{(\alpha\beta\gamma),\nu} \xi^\nu$$

or, with

$$\begin{aligned}\omega_{(\alpha)} &= h_{(\alpha)\nu} \xi^\nu, \\ \delta h_{(\alpha)\mu} &= - h_{(\alpha\beta\gamma)} h_\mu^{(\beta)} \omega^{(\gamma)} - \omega_{(\alpha),\mu}, \\ \delta h_{(\alpha\beta\gamma)} &= - h_{(\alpha\beta\gamma),\nu} h^{(\delta)\nu} \omega_{(\delta)}.\end{aligned}$$

It is well known that Lagrangian functions exist, which are densities and give Lagrangian equations which are equivalent to Einstein ones. These are functions of the tetrads and their first derivatives (and if one wishes, of the $h_{(\alpha\beta\gamma)}$), and one of the identities requires that the derivatives appear through the $h_{(\alpha)[\mu\nu]}$ only.

In the other identities appear the contravariant densities

$$f_{(\alpha)}^\nu \equiv \frac{\partial \mathcal{L}}{\partial h_{(\beta)\mu,\nu}} h_{(\beta\gamma\alpha)} h_\mu^{(\gamma)} - \mathcal{L} h_{(\alpha)}^\nu$$

and

$$f_{(\alpha)}^\nu + \frac{\delta \mathcal{L}}{\delta h_{\nu}^{(\alpha)}} \equiv \left(\frac{\partial \mathcal{L}}{\partial h_{\mu,\nu}^{(\alpha)}} \right)_{,\nu}.$$

One gets so conservation laws

$$\left(f_{(\alpha)}^\nu + \frac{\delta \mathcal{L}}{\delta h_{\nu}^{(\alpha)}} \right)_{,\nu} = 0$$

which are invariant for all changes of co-ordinates.

To put that in relation with Møller's complex, one has to consider

$$f_\lambda^\nu \equiv f_{(\alpha)}^\nu h_\lambda^{(\alpha)}; \quad f_\lambda^\nu \equiv \frac{\partial \mathcal{L}}{\partial h_{(\alpha)\mu,\nu}} h_{(\alpha)\mu,\lambda} - \mathcal{L} \delta_\lambda^\nu - \frac{\partial \mathcal{L}}{\partial h_{(\alpha)\mu,\nu}} h_{(\alpha)\lambda,\mu}.$$

In short: the last term makes the difference.

Les principes mécaniques de Galilée et la théorie d'Einstein.

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On doit à Galileo Galilée deux principes qui se sont trouvés de la plus grande importance pour le développement non seulement de la mécanique, mais de toute la physique.

C'est, en premier lieu, le principe de relativité de Galilée pour le mouvement rectiligne et uniforme. Dans un laboratoire en mouvement rectiligne et uniforme par rapport à un autre laboratoire fixe tout se passe comme dans cet autre laboratoire. Pour illustrer son principe, Galilée comparait les phénomènes physiques à l'intérieur d'un navire en repos à ceux dans un navire en mouvement uniforme et rectiligne; il parlait de navires et non de laboratoires. Mais les nombreux exemples de phénomènes physiques à l'intérieur d'un navire, cités par Galilée, nous montrent clairement qu'il avait en vue justement ce que nous appelons maintenant un laboratoire. La forme parfaitement physique et vérifiable par l'expérience de l'énoncé de Galilée mérite d'être accentuée.

Le deuxième principe peut être rapproché à l'observation faite par Galilée sur la chute des corps dans le vide: tous les corps, quel que soit leur poids, tombent avec la même vitesse s'ils n'éprouvent pas de résistance du milieu. Si l'on introduit, d'après Newton, la notion d'accélération, on peut formuler ce principe en disant que tous les corps qui tombent dans le vide éprouvent la même accélération.

Ces deux principes ont subi, à partir des temps de Galilée jusqu'à nos jours, un développement qui présente un grand intérêt.

D'abord, on a pu les préciser en s'appuyant sur les lois de mouvement établies par Newton. La première loi de Newton qui se rapporte, comme le premier principe de Galilée, au mouvement rectiligne et uniforme, indique le système de référence dans lequel ce principe est valable. C'est le système inertiel de Newton. En effet, la première loi de Newton peut être interprétée comme une définition d'un système inertiel. On peut la lire comme il suit: il existe un système de référence dans lequel un corps non soumis aux forces se meut d'une manière uniforme. C'est dans ce système de référence qu'on lie les autres lois de Newton. On

comprend alors pourquoi Newton a énoncé sa première loi de mouvement comme une loi à soi et non comme une conséquence de sa deuxième loi qui se rapporte au cas plus compliqué du mouvement sous l'action des forces. Le principe de relativité de Galilée est donc lié à la notion d'un système de référence inertiel.

D'autre part, le deuxième principe de Galilée, concernant la chute libre des corps, est lié aux notions de la masse inerte et de la masse pondérable ou pesante, notions qui ont, elles aussi, leur origine dans les idées de Newton. On peut énoncer ce principe comme égalité essentielle (ou équivalence) des deux notions de masse.

Trois siècles après Galilée, ses principes mécaniques ont subi une généralisation dans la théorie de relativité d'Einstein. Pour être exact, on doit parler de deux lignes de généralisation différentes, dont une seulement me semble complètement justifiable.

En premier lieu, le principe de relativité pour le mouvement rectiligne et uniforme a été étendu à tous les phénomènes physiques, y compris l'électromagnétisme et la propagation de la lumière. Dans sa théorie de relativité, établie en 1905, Einstein en a tiré des conséquences sur la nature des relations spatio-temporelles entre différents événements dans le monde physique, c'est à dire, sur la nature de l'espace et du temps. Il importe de souligner que les propriétés de l'espace et du temps ainsi trouvées, dont la plus importante est l'existence d'une valeur limite pour la vitesse de propagation de n'importe quelle action physique, caractérisent l'espace et le temps d'une manière absolue et non seulement relative à un observateur donné. Le relatif a conduit à l'absolu. Pour accentuer ce fait, on pourrait, suivant une heureuse proposition de Fokker, désigner la théorie d'Einstein comme chronogéométrie (et non comme théorie de relativité).

Donc, la première ligne de généralisation étend le principe de Galilée à tous les phénomènes physiques (et non seulement mécaniques), tout en se bornant au mouvement rectiligne et uniforme. Les équations de transformation pour les variables spatio-temporelles (les coordonnées cartésiennes et le temps) dans deux systèmes de référence en mouvement réciproque restent linéaires comme dans la transformation de Galilée, mais les coefficients y sont modifiés: la transformation de Galilée est remplacée par celle de Lorentz.

La deuxième ligne de généralisation suivie par Einstein a comme point de départ l'interprétation du second principe de Galilée comme principe d'équivalence entre l'accélération cinématique et la gravité, c'est à dire, entre les forces d'inertie et les forces de gravitation. Cette interprétation cinématique de la gravité qu'on nomme tout court « principe d'équivalence » a été combinée par Einstein avec le principe de relativité, et la fusion de ces deux principes a conduit Einstein à l'idée d'une « relativité générale ». Cette idée, assez vague et contestable, a suffi au génie

d'Einstein pour parvenir à sa théorie de gravitation qu'il a nommé théorie de relativité générale.

Disons tout de suite que la théorie de gravitation d'Einstein, qui est en même temps une théorie de l'espace et du temps, est une œuvre de génie. Elle s'impose par la beauté de sa conception et de sa réalisation mathématique. Elle donne la solution, attendue depuis longtemps, des problèmes posés par la théorie de Newton (par exemple, celui de l'action à distance). Elle admet des vérifications astronomiques et même expérimentales. Mais ce n'est pas une théorie de relativité générale, puisqu'une relativité générale n'existe pas.

Tout en acceptant sans réserve le résultat final d'Einstein, formulé au moyen de ses célèbres équations tensorielles et covariantes, nous allons voir si son raisonnement qui l'a conduit à ces équations est bien logique. Nous allons voir s'il est permis de parler de relativité générale et de l'équivalence complète entre les champs d'accélération et de gravitation, ou si plutôt un point de vue plus prudent est préférable qui est en même temps plus proche à celui de Galilée. Ce point de vue n'accepte le principe de relativité que pour le mouvement rectiligne et uniforme et utilise le principe d'égalité entre la masse inerte et la masse pondérable. En adoptant ce point de vue, nous devons constater que les concepts de relativité générale et d'équivalence ne sont pas, contrairement à l'opinion d'Einstein, les vrais principes de sa théorie. Nous devons donc trouver une réponse à la question suivante: quels sont les véritables principes qui forment la base de la théorie de gravitation d'Einstein?

Afin de trouver un point de départ solide pour notre analyse, essayons de formuler d'une manière plus précise ce que nous entendons par principe de relativité. Cela est d'autant plus nécessaire qu'on attache à ce terme des significations bien différentes; celle-ci se réduisent, en substance, à la signification physique et à la signification mathématique.

Nous adoptons le sens physique du terme et nous entendons par relativité *l'existence de phénomènes physiques correspondants* dans deux systèmes de référence (deux laboratoires). Cela veut dire qu'à *chaque* phénomène dans un système doit correspondre un phénomène identique dans un autre système. (Le sens du terme « identique » peut être précisé comme il suit: si le phénomène qui se passe dans chacun des deux systèmes-laboratoires est décrit par des fonctions ayant pour variables celles du système respectif, la forme mathématique de ces fonctions est identique pour les deux systèmes).

Dans les cas où les deux systèmes de référence sont inertiels, nous retrouvons le principe de relativité de Galilée dans sa forme précise. Il n'y a alors aucune difficulté de définir un système de référence associé à un laboratoire et l'on peut prendre comme tel les coordonnées cartésiennes et le temps. Dans les deux systèmes, ces variables sont liées par

la transformation de Lorentz. On voit alors la liaison entre la relativité au sens physique et le comportement des équations pour les transformations des coordonnées.

Par contre, si l'on tâche d'appliquer ces définitions aux systèmes non inertiels et aux mouvements non uniformes, c'est à dire, si l'on tâche de maintenir l'idée de relativité générale, on s'embrouille dans les difficultés.

Avant tout, pour les mouvements accélérés, le comportement d'un laboratoire, et même sa forme géométrique, dépend de la nature de ce laboratoire, de sorte qu'il devient impossible de définir un système-laboratoire d'une manière générale. Or, cette difficulté n'est pas la plus grande. Ce qui importe le plus, c'est le simple fait que le principe de relativité dans le sens propre du mot, c'est à dire le ~~principe~~ physique, est évidemment inapplicable aux systèmes accélérés. Pour le voir, il suffit de considérer une pendule ou une horloge à poids : une telle horloge peut très bien marcher dans un laboratoire terrestre (elle y peut servir comme instrument de grande précision), mais elle ne marche pas du tout sur un Spoutnik. Plus que cela : il n'y a pas de phénomène physique sur un Spoutnik qui pourrait correspondre à la marche d'une pendule sur la Terre. La prémisse fondamentale du principe de relativité — l'existence des phénomènes correspondants — n'est pas remplie dans ce cas.

Pour éviter ces difficultés et pour pouvoir appliquer le concept de relativité aux mouvements non uniformes, Einstein a introduit deux modifications essentielles, quoique voilées, dans l'usage des termes « système de référence » et « principe de relativité ». En premier lieu, il a modifié la définition d'un système de référence en le rapprochant non pas à un système physique (un laboratoire), mais à un système de coordonnées. Comme un changement de coordonnées non linéaire par rapport au temps entraîne l'introduction, dans les équations du mouvement, d'un champ de gravitation fictif, ainsi que des forces de Coriolis, cette nouvelle définition d'un système de référence signifie déjà, strictement parlant, un renoncement au principe de relativité physique. Ce principe a été « corrigé » par Einstein au moyen du principe d'équivalence tiré du deuxième principe mécanique de Galilée. Même dans le cas le plus simple du mouvement sans rotation, l'existence des phénomènes correspondants dans deux laboratoires n'est supposée qu'au prix d'introduction d'un champ de gravitation fictif dans un des laboratoires (ce champ fictif a pour but d'annuler, au besoin, un champ réel). Par exemple, le laboratoire réel à l'intérieur du Spoutnik est comparé non pas à un laboratoire réel sur la Terre, mais à un laboratoire fictif qui est immobile par rapport à la Terre, mais diffère du laboratoire terrestre réel par l'absence du champ de gravitation.

Revenons au principe d'équivalence. Il est incontestable que ce principe permet, dans le cas général, d'éliminer, dans une région d'espace

suffisamment petite (par exemple, à l'intérieur d'un Spoutnik) le champ de gravitation. Cependant, même si l'on néglige les forces électriques qui ne se prêtent pas à une élimination analogue et qui peuvent indiquer, en principe, l'existence de l'accélération et de la rotation absolue, le recours au principe d'équivalence ne peut pas sauver l'idée de la relativité générale. Pour pouvoir parler de relativité générale, Einstein a été obligé de priver ce terme de son sens physique et de lui attacher un sens tout différent qui est purement mathématique. Il lui a d'abord donné l'interprétation un peu vague de « la même forme des lois de nature » dans différents systèmes de référence; puis il a remplacé les systèmes de références par les systèmes de coordonnées et les lois de nature par les équations différentielles du champ ou du mouvement (qui cependant, à elles seules, ne déterminent pas le cours d'un phénomène physique). En raisonnant ainsi, Einstein est parvenu à interpréter la « relativité générale » comme « covariance des équations différentielles pour un changement arbitraire de coordonnées ». Mais cette interprétation est tout à fait différente du principe de relativité physique. La nouvelle interprétation est une exigence purement logique et non pas physique, exigence qui doit être remplie même si le principe de relativité physique n'existe pas. En effet, tant qu'on ne précise pas le système de coordonnées, il faut bien que les équations différentielles écrites dans différents systèmes soient équivalentes. On connaît d'ailleurs depuis longtemps un exemple du formalisme qui satisfait à cette exigence: ce sont les équations de Lagrange de deuxième espèce.

En attaquant l'idée de relativité générale et en insistant sur le caractère limité du principe d'équivalence, nous sommes loin de nier la valeur euristique de ces principes et leur rôle historique dans la création de la théorie de gravitation par Einstein. Quelles que soient les lacunes logiques dans le raisonnement d'Einstein, on doit admettre que les principes de relativité et d'équivalence lui ont indiqué l'avantage qu'on peut tirer du rapprochement des relations cinématiques spatio-temporelles et de la gravitation, et lui ont aidé à trouver la forme mathématique, en même temps simple et covariante, de ses équations gravifiques.

Mais en admettant tout cela, nous devons constater que ces deux principes ne sont pas la vraie base logique de la théorie d'Einstein. Il nous faut donc chercher d'autres principes qu'on pourrait considérer comme formant la base de sa théorie. Puisque Einstein nous a donné la formulation mathématique complète de sa théorie de gravitation, il n'est pas difficile d'indiquer ces vrais principes.

La première idée fondamentale est l'unification de l'espace et du temps en une seule entité chrono-géométrique à quatre dimensions dont une est temporelle et les trois autres spatiales. On exprime cela en disant que la métrique de cette entité est indéfinie de signature $(+, -, -, -)$. La

forme de la métrique est liée à la loi de propagation d'une action quelconque, par exemple du front d'onde lumineuse, avec la vitesse-limite. Cette vitesse-limite est toujours égale à celle de la lumière. L'existence d'une vitesse-limite est un fait fondamental du monde physique qui a profondément changé nos notions sur l'espace et le temps. Remarquons que, d'après Robb et Alexandrov, ce fait peut être rapproché à la notion de cause et effet.

L'idée de l'unification de l'espace et du temps a été réalisée par Einstein dans les deux formes de sa théorie de relativité : celle de 1905, dite relativité restreinte, est celle de 1916, dite relativité générale. Dans la théorie de 1905, la métrique de l'espace-temps admet pour l'intervalle infiniment petit une expression à coefficients constants, et dans la théorie de 1916 la métrique est supposée plus générale, à savoir, riemannienne (ce sont justement les coefficients de l'expression pour l'intervalle — les composantes du tenseur métrique — qui la caractérisent).

La deuxième idée fondamentale de la théorie de gravitation d'Einstein, celle qui la distingue de la théorie de relativité dite restreinte, est l'hypothèse que la métrique de l'espace-temps n'est pas donnée une fois pour toutes, qu'elle n'est pas « rigide », mais qu'elle peut dépendre de ce qui se passe dans l'espace-temps, en premier lieu, de la distribution et du mouvement des masses. Cette idée, conçue par Einstein entre 1908 et 1915 (probablement en 1911, pendant son séjour à Prague) est entièrement nouvelle — Einstein n'a pas eu ici de prédécesseurs. La portée de cette idée est immense. En s'appuyant sur cette idée, Einstein a pu établir le lien entre la métrique et le phénomène de gravitation, un lien si étroit qu'on peut parler d'unité. Cette unité se traduit par le fait que les composantes du tenseur métrique définissent complètement le champ de gravitation, de sorte qu'il devient inutile d'introduire d'autres fonctions pour caractériser ce champ. On sait qu'en partant de cette hypothèse et en s'appuyant sur une certaine analogie avec les équations de gravitation newtoniennes on parvient aux équations d'Einstein d'une manière presque unique.

Nous avons vu comment les principes mécaniques de Galilée, interprétés par Einstein d'une manière pas toujours logique, lui ont aidé à formuler sa célèbre théorie de gravitation, dont nous avons essayé à résumer les deux idées fondamentales. Nous allons maintenant examiner la relation entre les principes de Galilée et la théorie d'Einstein d'une manière différente. Nous allons poser la question : étant donnée la théorie d'Einstein dans sa forme définitive, quelle est la place occupée dans cette théorie par les principes de Galilée ?

On pourrait croire, de premier abord, que le principe de Galilée relatif au mouvement rectiligne et uniforme n'y joue aucun rôle, puisqu'en présence des masses et des champs de gravitation l'espace-temps cesse

d'être homogène (l'expression pour l'intervalle n'est pas réductible aux coefficients constants). Cependant, il n'en est rien. En effet, pour un système de masses du type du système solaire, on peut imposer au tenseur métrique quatre équations supplémentaires qui sont compatibles avec les équations d'Einstein et qui expriment une condition auxiliaire pour les coordonnées. Cette condition exige que chacune des coordonnées spatiales et la variable temporelle doivent satisfaire à l'équation d'onde (équation généralisée de d'Alembert). En supposant remplies certaines conditions à l'infini pour le tenseur métrique on peut affirmer que ces quatre variables, qu'on appelle « coordonnées harmoniques », sont définies à une transformation de Lorentz près. Dans la théorie de gravitation d'Einstein, le système de coordonnées harmoniques est l'analogue le plus proche des coordonnées cartésiennes et du temps associées à un système inertiel de Newton. Puisque les transformations admissibles pour ces coordonnées sont linéaires, le mouvement rectiligne et uniforme conserve sa position privilégiée qu'il occupait dans l'énoncé du principe de relativité de Galilée. Ce principe reste donc en vigueur comme principe physique dans le cas considéré de l'espace inhomogène d'Einstein. Il va sans dire qu'en définissant les phénomènes physiques correspondants dans deux systèmes de référence on doit y inclure le mouvement correspondant des masses gravifiques. Remarquons que la possibilité de définir les phénomènes physiques correspondant dans un espace-temps non homogène n'existe que grâce à la non-rigidité de la métrique.

On voit que, pour une classe très-étendue de problèmes (pour tous les problèmes non cosmologiques), le principe de relativité de Galilée reste en vigueur, tandis que la relativité générale au sens physique n'existe pas.

Quant au second principe de Galilée interprété dans le sens de Newton comme égalité entre la masse inerte et la masse pondérable, il reste en vigueur, pourvu qu'une définition séparée des deux qualités de la masse est possible. Dans la théorie d'Einstein, une telle définition n'a qu'une valeur approximative. Mais, puisque dans les solutions des équations d'Einstein, une même constante intervient comme masse inerte et masse pondérable, on peut dire que les deux masses sont égales tant que ces solutions sont valables. Or, l'application du second principe de Galilée interprété comme égalité des deux masses ne se borne pas à une petite région d'espace, contrairement au principe d'équivalence qui est de nature strictement locale.

Les principes mécaniques de Galilée que nous avons analysé ne constituent qu'une partie de son œuvre gigantesque dont les conséquences pour la physique peuvent être poursuivies jusqu'à nos jours. Mais notre analyse, si courte et incomplète qu'elle soit, suffit déjà pour entrevoir l'influence profonde de son œuvre sur la pensée humaine — influence qui dure des siècles.

INTERVENTI E DISCUSSIONI

— M. A. TONNELAT:

Si l'on replace les Principes Mécaniques cités par le Professeur Fock dans les perspectives de l'époque galiléenne, il peut être utile de les formuler un peu différemment.

Principe d'inertie.

Le mouvement réel d'une masse *réelle* et libre — c'est à dire nécessairement pourvue d'une gravité intrinsèque — s'effectue, selon Galilée, sur des surfaces qui seraient pour nous des surfaces équipotentiels de gravité (l'expression n'est évidemment pas dans Galilée mais l'idée y est très exactement explicitée). Pour le physicien qu'est Galilée le mouvement rectiligne et uniforme d'un point matériel libre mais réel, est une notion asymptotique. C'est en ce sens qu'on peut soutenir que Galilée, physicien et non essentiellement philosophe comme Descartes, fut plus proche des conceptions propres à la Relativité Générale que de celles de la Restreinte.

Principe de Relativité.

Pour Galilée, l'idée d'une indépendance des phénomènes vis à vis du mouvement rectiligne et uniforme du système de référence est tout à fait générale et s'étend implicitement aux phénomènes lumineux.

D'ailleurs jusqu'à la fin du XIX^e siècle il va de soi que la vitesse de la lumière, comme celle des projectiles matériels, n'est pas influencée par le mouvement du système lié à la source (problème des boulets de canon tirés vers l'ouest ou vers l'est).

La formule galiléenne de composition des vitesses

$$c' = c + v \quad (n = 1, v \neq 0)$$

exprime de la même façon la relativité du mouvement de la matière et de la lumière.

C'est au XIX^e siècle seulement que l'*expérience* (propagation dans les milieux réfringents; électrodynamique des corps en mouvement) exige une formule de composition:

$$c' = \frac{c}{n} + \alpha v \quad \left(\alpha = 1 - \frac{1}{n^2} \right) \rightarrow c' = c \quad (n = 1, v \neq 0),$$

en contradiction avec la Relativité Galiléenne. C'est alors, mais alors seulement, que semble surgir le caractère absolu (au sens de Galilée) des phénomènes lumineux. Au temps de Galilée ou même de Newton la question ne se pose pas.

— H. J. TREDER:

From the mathematical point of view the Einstein special principle of relativity is not more general as the Galilei principle. But the Einstein prin-

ciple means the invariance in reference to a other space-time-group as the Galilei principle means.

The Galilei-group (including the threedimensional spatial group) is different from the Lorentz-group by the nonexistence of a maximal velocity. The Lorentz-group is degenerated to the Galilei-group if the maximal velocity becomes infinite.

The Galilei-group is the inhomogeneous pseudo-orthogonal group for a four-dimensional space-time with a isotropic dual metric.

That is proof in a paper by Friedrichs (1927). According to Friedrichs the general Galilei-group describes the 10 symmetries of a degenerate flat space-time whereas the Lorentz-group describes a nondegenerate space-time. The physical cause of the degeneration is the possibility of infinite velocities in the Galileian mechanics.

— S. DESER:

Professor Fock, in his insistence on separating the notion of covariance from that of the corresponding « gauge » field, has greatly clarified the mathematical role of invariance requirements in general relativity. Indeed, a simple parallel exists in electrodynamics; one could have « gauge » invariance of the second kind for charged fields, without physical photons. His emphasis on the fact that boundary conditions are an integral part of a theory, like its differential equations, is also very salutary; certainly we are closer to the other laws of physics when incorporating asymptotic flatness into the theory.

However, just because Fock insists on the importance of inertial frames where they are defined (in the absence of gravitation), it seems puzzling that he singles out harmonic frames globally rather than merely asymptotically. In the interior where there is gravitation, no physically preferred inertial frame exists. At infinity, on frames reducing to Lorentzian ones, satisfy the special relativity requirement, and this includes the wide variety (beside harmonic ones). In fact, from considerations related to the quantization problem gauges or frames which are three-dimensional (« radiation » gauges) may well be physically superior to four-dimensional « Lorentz » gauges.

— C. CATTANEO:

M. Fock a souligné que dans la théorie einsteinienne restreinte il y a une véritable relativité physique tandis que dans la théorie générale il n'y a que la covariance des équations. On pourrait peut être préciser que si dans la théorie restreinte il y a une relativité physique *actuelle* (bornée à la classe des repères galiléens), dans la relativité générale il y a une relativité plus étendue, mais seulement potentielle: même dans la théorie générale on aurait complète équivalence physique en deux repères différents si par hasard la distribution et le mouvement des masses de l'univers étaient les mêmes par rapport aux deux repères considérés.

— C. MÖLLER:

In connection with Prof. Cattaneo's remarks, with which I entirely agree, I would like to make a few supplementary remarks. The fact that a pendulum

clock works well on the earth but not in a sputnik is, in my opinion, no proof that the general principle of relativity is not valid, as maintained by Professor Fock. It only shows that we have a gravitational field on the earth but not in the sputnik. The situation is here quite analogous to the case of an electromagnetic field in the special theory of relativity, where one may have a magnetic field in one system of inertia but no magnetic field in the rest system of the charge producing the field. In the latter case it would certainly not be right to conclude that the special principle of relativity is not valid.

— V. A. FOCK:

La constatation faite par M. Møller que les conditions physique dans un Spoutnik sont différentes de celles dans un laboratoire terrestre est équivalente à la constatation qu'il n'y a pas de relativité physique pour les deux systèmes.

Relativité physique signifie justement identité des conditions physiques.

— J. A. WHEELER:

In connection with the illuminating analysis by Professor Fock of the history of ideas from Galileo to Einstein, it is appropriate to qualify his remark « l'hypothèse que la métrique... puit dependre de ce qui se passe..., conçue par Einstein... probablement en 1911..., est entièrement nouvelle ».

Here is what Einstein himself says (A. EINSTEIN: *Essays in Science*, Philosophical Library, (New York, 1934), p. 68): « Physical space and the ether are different names for the same thing; fields are physical conditions of space.... But... physicists were still far removed from such a way of thinkings space was still, for them, a rigid homogeneous something, susceptible of no change or conditions. Only the genius of Riemann, solitary and uncomprehended, had already won its way by the middle of the last century to a new conception of space, in which space was deprived of its rigidity, and in which its power to take part in physical events was recognized as possible ». It is particularly appropriate to recall this pioneering contribution of Riemann at a meeting held in Italy. Riemann loved Italy. Knowing as a young man, afflicted by tuberculosis, that he was soon to die, he came to Italy to do his last two pieces of work: one, an attempt at a unified theory of gravitation and electromagnetism; the other, a joint work with the Italian mathematician Betti on the classification of topologies. Today we have learned that Riemann's ideas on *topology* have physical consequences comparable in interest to those which Einstein found in Riemann's *geometry*. Einstein taught us that a *local* property of space—the Riemann curvature—is to be identified with a local feature of the gravitation field—the so-called tide-producing force. More recently we have learned that a *global* property of space—the number of electric lines of force trapped in one « worm-hole » or « handle » of the topology—is the classical counterpart of what, in the real world of quantum physics, we call electric charge (J. A. WHEELER: *Geometrodynamics* (New York, 1962)).

The Betti number of the topology, that is the number of « wormholes » tells us the number of independent charge-like magnitudes that are required to specify the state of the electromagnetic field.

Thus, we owe to Riemann's vision not only the idea that geometry is a branch of physics, but also the even more recently appreciated concepts of *physics as a manifestation of geometry-plus-topology*.

— H. BONDI:

I agree with Fock that the general principle of relativity is empty. We know of course that there is no *physical* equivalence between inertial and accelerate observers. (A passenger in a car knows whether the road is full of holes without seeing the holes). I feel confident that, given any laws, mathematicians could find a way of writing these laws in a *mathematically* equivalent way. Secondly, as the Michelson-Morley experiment has been mentioned, I want to stress that its importance, though great, is purely historical. A modern physicist, like his predecessors, would want to keep the distances to the mirrors accurately constant, but with modern technology he might try to achieve this by having the mirrors moved by servomotors activated by a distance-measuring device based on interference fringes. If the mirrors are moved to keep the interference fringes in constant position, of course the fringes will not move, and so we know the outcome of the experiment without performing it, *i.e.* it is not worth performing it.

My third point concerns the foundations of general relativity. Applying Galileo's result, that all bodies fall equally fast, to Newton's theory of gravitation, we find that the only locally measurable quantity is the relative acceleration of neighbouring particles. Therefore neither V (the Newtonian potential) nor $V_{,i}$ are locally measurable but only $V_{,ii}$, for slowly moving particles with the local application of special relativity, we can arrive easily at least at the form of the equation of geodesic deviation and so introduce the curvature tensor from dynamics.

— P. G. BERGMANN:

Prof. Fock has stated that the group of transformations between harmonic co-ordinates is isomorphic to the Lorentz group. Has anyone ever proved this statement?

— V. A. FOCK:

The statement has been proved in the approximation used when deriving the equations of motion (including second order relativistic corrections). Also it is proved rigorously for flat space. A rigorous and general proof is still to be found.

— P. G. BERGMANN:

There are atomic and nuclear clocks in noninertial systems. (Harwell-Mossbauer experiment).

The principle of equivalence refers to *first-order* derivatives of the gravitational potentials.

— A. LICHTNEROWICZ:

Je voudrais dire mon complet accord avec le point de vue du Prof. Fock concernant la relativité générale (sauf peut-être en ce qui concerne l'existence de coordonnées harmoniques *globales*).

C'est pour moi un non-sens de faire jouer le rôle d'un principe en relativité générale à la covariance des équations. Toute théorie physique peut être mise sous une forme indépendante de tout repérage de l'espace temps (c'est le cas de la mécanique classique).

Je voudrais seulement remarquer que ce sont les représentations d'un groupe fondamental qui jouent ici le rôle essentiel (en relativité générale le groupe de Lorentz). Nous les réalisons à l'aide d'objets géométriques (tenseurs ou spineurs) représentés à partir de repères. Dans le cas tensoriel, cela peut être des repères linéaires dans le cas spinoriel, il convient que ce soit des repères orthonomés.

— V. A. FOCK :

Je peux résumer ma thèse principale en deux courtes phrases :

- 1) la relativité physique n'est pas générale;
- 2) la relativité générale n'est pas physique.

— M. MERCIER :

J'aimerais ajouter deux remarques : l'une en complément du rappel historique que faisait J. Wheeler à propos de Riemann. Si nous savons tous que Riemann avait déjà réfléchi à la possibilité d'une relation entre la géométrie et la gravitation, il semble moins connu que Clifford y avait également pensé, et que par conséquent il faut placer ce mathématicien anglais parmi les précurseurs.

La seconde remarque est que si, comme l'a fort bien relevé M. Fock, Einstein a postulé que la géométrie—c'est à dire ici la métrique—dépend de la répartition et du mouvement de la matière, il n'est plus possible dans sa théorie de dire dans quel sens va cette relation : la répartition de la matière dépend de la métrique tout autant que la métrique dépend de la répartition de la matière. Il s'en suit un déterminisme « absolu » en ce sens que plus rien ne peut être donné pour en tirer quelque chose comme des prévisions. C'est une situation à l'opposé de la conception newtonienne, rendant l'attaque des problèmes de la pratique physique très délicate, en un sens aléatoire. On comprend que un savant qui a établi les règles d'un déterminisme pareil, ait toujours eu de la peine à se rallier au déterminisme conditionnel de la physique quantique.

The Motion of Multipoles in General Relativity.

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1. - Introduction.

It is the purpose of this paper to discuss the motion of quadrupole test particles in general relativity. We shall use the technique of using a pair of Schwartz distributions to describe such particles as was done in the case of dipole particles [1]. In that case it was shown that a condition which ruled out the existence of mass dipoles had two effects: *a)* a supplementary condition which competed the differential equations describing the motion of the dipole followed from it and *b)* this supplementary condition implied that the «mass» of the dipole was a constant.

It will be shown below that the theory of quadrupole particles differs from that of monopoles and dipoles in that the differential equations describing them involve a tensor, analogous to the quadrupole moment tensor, whose behavior along the world line of the particle is not derivable from the equations of motion of the particle. It will be further shown that the «mass» of the quadrupole is not a constant along the world line of the particle but its variation along this curve is related to the variation of the tensor mentioned above.

Such a tensor arises in the discussion of the perturbation of the gravitational field due to a perturbation in the stress-energy tensor. Thus the theory presented here relates the behavior of the «mass» of a quadrupole with the behavior of a tensor which can be related to the gravitational energy radiated by such a test particle.

A quadrupole test particle is characterized by a tensor valued and a scalar valued distributions, $T^{\mu\nu}$ and ϱ , such that

$$(1.1) \quad U^e \int T^{\mu\nu} f_{\mu\nu} n_e dv = M^{\mu\nu} f_{\mu\nu} - M^{\mu\nu\sigma} f_{\mu\nu;\sigma} + M^{\mu\nu\sigma\tau} f_{\mu\nu;\sigma\tau}$$

and

$$(1.2) \quad U^e \int \varrho u^\mu f_\mu n_e dv = N^\mu f_\mu - N^{\mu\sigma} f_{\mu;\sigma} + N^{\mu\sigma\tau} f_{\mu;\sigma\tau}$$

where $f_{\mu\nu}$ is an arbitrary symmetric tensor, f_μ is an arbitrary vector, n_e is the normal to a three-dimensional hypersurface which we shall take to be orthogonal to a world-line $x^\mu = X^\mu(s)$, the support of the distributions, whose tangent vector is

$$U^\mu = \frac{dX^\mu}{ds},$$

u_μ is a normalized time-like proper vector of the stress-energy tensor $T^{\mu\nu}$, that is,

$$(1.3) \quad g_{\mu\nu} u^\mu u^\nu = 1,$$

and

$$T^{\mu\nu} u_\nu = \varrho(1 + \varepsilon/c^2)u^\mu,$$

the integrals on the left hand sides of eqs. (1.1) and (1.2) are taken over the hypersurface whose normal is n_e and whose invariant volume measure is $n_e dV$, and the right hand sides of eqs. (1.1) and (1.2) involve only quantities evaluated on the world line $x^\mu = X^\mu(s)$. The tensors $M^{\mu\nu}$, $M^{\mu\nu\sigma}$, $M^{\mu\nu\sigma\tau}$, N^μ , $N^{\mu\sigma}$, $N^{\mu\sigma\tau}$ then characterize the quadrupole. They satisfy the conditions

$$(1.5) \quad \begin{cases} M^{\mu\nu\sigma} = M^{\nu\mu\sigma}, \\ M^{\mu\nu\sigma\tau} = M^{\mu\nu\alpha\beta} P_\alpha^\sigma P_\beta^\tau = M^{\mu\nu\sigma\tau} = M^{\mu\nu\tau\sigma}, \\ N^{\mu\sigma\tau} = N^{\mu\alpha\beta} P_\alpha^\sigma P_\beta^\tau = N^{\mu\tau\sigma}, \end{cases}$$

where

$$(1.6) \quad P_\alpha^\sigma = \delta_\alpha^\sigma - \frac{u^\sigma n_\alpha}{u^e n_e}$$

and is the projection tensor onto the hypersurface with normal n_α . For convenience we shall assume that $n_\alpha = k u_\alpha$ and hence

$$(1.7) \quad P_\alpha^\sigma = \delta_\alpha^\sigma - u^\sigma u_\alpha.$$

This choice is consistent with the requirement that the hypersurface entering into eqs. (1.1) and (1.2) be orthogonal to the world line which is the support of the distribution and which will be called the world line of the particle.

The notations and terminology used above are similar to those used in ref. [1]. If we assume that

$$M^{\mu\nu\sigma\tau} = N^{\mu\sigma} = 0$$

the theory given below reduces to that of the dipole. If in addition we assume that

$$M^{\mu\nu\sigma} = N^{\mu\sigma} = 0$$

we obtain the theory of the monopole.

2. - The conservation of energy and momentum.

The stress-energy tensor is usually assumed to satisfy the equations

$$T^{\mu\nu}_{;\nu} = 0$$

which are equivalent to the equation

$$(T^{\mu\nu} \lambda_\nu)_{;\mu} = T^{\mu\nu} \lambda_{\mu;\nu}$$

for arbitrary vector fields λ_μ , where the semicolon denotes the covariant derivative with respect to the space-time over which $T^{\mu\nu}$ is defined. We may write the latter equation as

$$(2.1) \quad \int T^{\mu\nu} \lambda_\mu n_\nu dV = \int T^{\mu\nu} \lambda_{\mu;\nu} \sqrt{-g} d^4x$$

where the right hand integral is evaluated over a four-dimensional region in space time and the left hand integral is taken over a closed hypersurface bounding this region with normal vector n_ν .

Equation (2.1) will be assumed to hold when $T^{\mu\nu}$ is a tensor valued distribution. The methods used in ref. [1] may then be applied to deduce from this equation and equation (1.1) the equation

$$(2.2) \quad \frac{d}{ds} [P^q \lambda_q - L^{q\sigma} \lambda_{q;\sigma} + I^{q\sigma\tau} \lambda_{q;\sigma\tau}] = M^{q\sigma} \lambda_{q;\sigma} - M^{q\sigma\tau} \lambda_{q;\sigma\tau} + M^{q\sigma\tau\lambda}_{\lambda_{q;\sigma\tau}}$$

where in a Gaussian co-ordinate system based on the hypersurface used in eq. (1.1),

$$(2.3) \quad U^4 P^q = M^{4q} + I^{\alpha\beta} M^{\alpha q\beta} + 2 \Gamma^4_{\alpha\beta} \Gamma^\alpha_{\sigma\tau} M^{q\sigma\tau} - \Gamma^4_{\alpha\beta,\gamma} M^{q\alpha\beta\gamma}$$

and the comma denotes the ordinary derivative;

$$(2.4) \quad U^4 L^{e\sigma} = M^{4e\sigma} + 2\Gamma_{\alpha\beta}^4 M^{e\alpha\beta\sigma};$$

$$(2.5) \quad U^4 I^{\nu\sigma\tau} = M^{4\nu\sigma\tau}.$$

It can be shown that P^e , $L^{e\sigma}$ and $I^{e\sigma\tau}$ are tensor quantities defined along the world-line of the particle. The vector P^e occurs in the special theory of relativity in the discussion of an isolated system without internal motions. In that case it is called the energy momentum vector of the system. It is reasonable to assign that interpretation to this vector for the case being considered here and to define the mass of the quadrupole as the quantity

$$(2.6) \quad m = P^e U_e.$$

The equations of motion of the quadrupole are obtained from eq. (2.2) by using the fact that these equations must hold for arbitrary vector fields λ_μ and hence the coefficients of the various derivatives of λ_μ must be equal on both sides of this equation. We thus obtain the equations

$$(2.7) \quad M^{\nu e\sigma\tau} - U^e I^{\nu\sigma\tau} + M^{\nu\sigma\tau e} - U^\sigma I^{\nu\tau e} + M^{\nu\tau e\sigma} - U^\tau I^{\nu e\sigma} = 0$$

$$(2.8) \quad M^{e\sigma\tau} + M^{e\tau\sigma} - L^{e\sigma} U^\tau - L^{e\tau} U^\sigma + 2 \frac{D I^{e\sigma\tau}}{Ds} = 0$$

$$(2.9) \quad P^e U^\sigma - \frac{D L^{e\sigma}}{Ds} - M^{e\sigma} + R^e_{\nu\alpha\beta} U^\beta I^{\sigma\nu\alpha} + R^\sigma_{\nu\alpha\beta} U^\beta I^{e\nu\alpha} + \\ + R^e_{\nu\beta\alpha} M^{\nu\alpha\beta\sigma} + \frac{1}{8} R^\sigma_{\alpha\beta\gamma} P^\gamma M^{e\nu\alpha\beta} = 0$$

and

$$(2.10) \quad \frac{D P^e}{Ds} + \frac{1}{2} (M^{\mu\sigma\tau} - L^{\mu\sigma} U^\tau) R^e_{\mu\sigma\tau} - I^{\nu\alpha\sigma} U^\tau R^e_{\nu\tau\alpha;\sigma} + \frac{2}{3} M^{\nu\sigma\tau} P^\alpha_\gamma R^e_{\nu\tau\alpha;\sigma} = 0$$

where the symbol D/Ds represents the absolute derivative along the world time of the particle. For example

$$\frac{D}{Ds} L^{e\sigma} = \frac{d L^{e\sigma}}{ds} + L^{e\alpha} \Gamma_{\alpha\beta}^\sigma U^\beta + L^{\alpha\sigma} \Gamma_{\alpha\beta}^e U^\beta.$$

In the derivation of these equations use was made of facts such as

$$\frac{d}{ds} (P^\sigma \lambda_e) = \frac{D P^\sigma}{Ds} \lambda_\sigma + \lambda_{\sigma;e} U^e$$

and eqs. (2.7) were used to simplify subsequent equations.

If in these equations we set $M^{\mu\nu\sigma\tau} = 0$ we obtain the equations of the dipole given in ref. [1].

3. - Equations (2.7).

It may readily be verified by using a rest system, a co-ordinate system at the point with parameter value s on the world line of the particle, in which the vector $U^\sigma = \delta_4^\sigma$, that these equations together with eqs. (15), constitute a system of 40 linear equations for 60 unknowns. They may be written in a general co-ordinate system as

$$(3.1) \quad P_\alpha^\epsilon P_\beta^\sigma P_\gamma^\tau (M^{\mu\alpha\beta\gamma} + M^{\mu\beta\gamma\alpha} + M^{\mu\gamma\alpha\beta}) = 0 .$$

Their general solution is given by

$$(3.2) \quad M^{\mu\alpha\sigma\tau} = P_\alpha^\sigma P_\beta^\tau (Q_1^{\alpha\beta} U^\mu + Q_1^{\mu\alpha\beta} U^\epsilon)$$

where

$$Q_1^{\alpha\beta} = Q_1^{\beta\alpha}$$

and

$$Q_1^{\alpha\beta\gamma} + Q_1^{\beta\gamma\alpha} + Q_1^{\gamma\alpha\beta} = 0 ,$$

that is

$$Q_1^{\alpha\beta\gamma} = Q^{\alpha\beta\gamma} - \frac{1}{3}(Q^{\alpha\beta\gamma} + Q^{\beta\gamma\alpha} + Q^{\gamma\alpha\beta}) .$$

with

$$Q^{\alpha\beta\gamma} = Q^{\alpha\gamma\beta} .$$

The ensuing discussion is greatly simplified if we assume that

$$(3.3) \quad P_\alpha^\sigma P_\beta^\tau P_\gamma^\epsilon Q_1^{\alpha\beta\gamma} = 0 .$$

This condition need not be imposed but by making it we simplify many calculations and preserve the features of quadrupoles we wish to discuss.

It follows from eqs. (3.2), (3.3) and (1.7) that

$$P_\epsilon^\lambda M^{\mu\epsilon\sigma\tau} = 0 .$$

Since

$$M^{\mu\epsilon\sigma\tau} = I^{\epsilon\sigma\tau} U^\mu + P_\lambda^\mu M^{\lambda\epsilon\sigma\tau}$$

where

$$I^{\varrho\sigma\tau} = U_{\mu} M^{\mu\varrho\sigma\tau},$$

it follows that

$$(3.4) \quad M^{\mu\varrho\sigma\tau} = U^{\mu} I^{\varrho\sigma\tau}$$

Similarly we may write

$$(3.5) \quad I^{\varrho\sigma\tau} = U^{\varrho} I^{\sigma\tau}$$

where

$$(3.6) \quad I^{\sigma\tau} = I^{\tau\sigma} = P_{\lambda}^{\sigma} P_{\mu}^{\tau} I^{\lambda\mu}$$

and hence has nine independent components.

The tensor $I^{\sigma\tau}$ is defined in terms of the integral properties of the stress-energy tensor. When it satisfies the additional property

$$g_{\sigma\tau} I^{\sigma\tau} = 0,$$

it may be shown to be related to the tensor

$$D^{ij} = \int T^{44} \left(x^i x^j - \frac{1}{3} \Sigma (x^i)^2 \delta^{ij} \right) d^3x, \quad i, j = 1, 2, 3$$

entering into approximate the solution of the Einstein field equations (cf. ref. [2]).

4. - Equations (2.8).

When these equations are multiplied by $P_{\sigma}^{\mu} P_{\tau}^{\nu}$ and summed they become

$$(4.1) \quad M^{\varrho\mu\nu} + M^{\varrho\nu\mu} = U^{\mu} U_{\sigma} M^{\sigma\varrho\nu} + U^{\nu} U_{\sigma} M^{\sigma\varrho\mu} - U^{\mu} U^{\nu} F^{\varrho} - 2\dot{I}^{*\varrho\mu\nu}$$

where

$$\dot{I}^{*\varrho\mu\nu} = \left(\frac{DI^{\varrho\sigma\tau}}{Ds} \right) P_{\sigma}^{\mu} P_{\tau}^{\nu}$$

$$F^{\varrho} = U_{\sigma} U_{\tau} M^{\sigma\varrho\tau}$$

If the indices $\varrho\mu\nu$ are cyclically permuted to $\mu\nu\varrho$ and $\nu\varrho\mu$ and if the first of the ensuing equations is added to (4.1) and the second subtracted from the result we then obtain

$$(4.2) \quad 2M^{\varrho\mu\nu} = U^{\mu} S^{\varrho\nu} + U^{\varrho} S^{\mu\nu} + U^{\nu} (U_{\sigma} M^{\sigma\varrho\mu} + U_{\sigma} M^{\sigma\mu\varrho}) -$$

$$- U^{\mu} U^{\nu} F^{\varrho} - U^{\nu} U_{\sigma} F^{\mu} + U^{\varrho} U^{\mu} F^{\nu} - 2\dot{I}^{*\varrho\mu\nu} - 2\dot{I}^{*\mu\nu\varrho} + 2\dot{I}^{*\nu\varrho\mu}$$

where

$$(4.3) \quad S^{e\sigma} = U_\sigma (M^{\sigma e\sigma} - M^{\sigma\sigma e})$$

On multiplying eqs. (4.1) by U_e and substituting into eqs. (4.2) we obtain

$$(4.4) \quad 2 M^{e\sigma\tau} = P_\alpha^\tau (U^\sigma S^{e\alpha} + U^e S^{\sigma\alpha} + 2 \dot{I}^{*\alpha e\sigma} - 2 \dot{I}^{*\sigma e\alpha} - 2 \dot{I}^{*\sigma\alpha e})$$

since

$$(4.5) \quad M^{e\sigma\tau} U_\tau = 0 .$$

In view of eqs. (3.5) and (3.6) we may write eqs. (4.4) as

$$M^{e\sigma\tau} = \frac{1}{2} P_\alpha^\tau (U^\sigma S^{e\alpha} + U^e S^{\sigma\alpha}) + \dot{U}^\tau I^{e\sigma} - \dot{U}^e I^{\sigma\tau} - \dot{U}^\sigma I^{e\tau} - U^e \dot{I}^{*\sigma\tau} - U^\sigma \dot{I}^{*e\tau}$$

where we have used the dot to represent the operation D/Ds and defined

$$(4.6) \quad \dot{I}^{*\sigma\tau} = \dot{I}^{\alpha\beta} P_\alpha^\sigma P_\beta^\tau .$$

It is a consequence of eqs. (4.6) that

$$(4.7) \quad U_e M^{e\sigma\tau} = \frac{1}{2} [U^\sigma U_e S^{e\tau} + U^\tau U_e S^{e\sigma} + S^{\sigma\tau}] - \dot{I}^{*\sigma\tau} .$$

In the Gaussian co-ordinate system in which the equations given above hold we have $n_\alpha = \delta_\alpha^4 = U_\alpha$ and eqs. (2.4) become

$$L^{e\sigma} = U_\alpha M^{\alpha e\sigma} - 2 \dot{U}_\beta I^{\beta\sigma} U^e$$

or

$$(4.8) \quad L^{e\sigma} = U_\alpha M^{\alpha e\sigma} + 2 U^e \dot{I}^{\beta\sigma} U_\beta$$

where use has been made of eqs. (3.4), (3.5) and the fact that

$$I^{\sigma\tau} U_\tau = 0 .$$

Note that it is a consequence of this equation that

$$\dot{I}^{\sigma\tau} U_\sigma U_\tau = 0 .$$

It then follows from eqs. (4.8) and (4.9) that

$$(4.9) \quad L^{e\sigma} = \frac{1}{2} [U^\sigma U_\alpha S^{\alpha e} + U^e U_\alpha S^{\alpha\sigma} + S^{e\sigma}] - \dot{I}^{e\sigma} + 3 U^e \dot{I}^{\beta\sigma} U_\beta + U^\sigma \dot{I}^{\beta e} U_\beta .$$

Hence

$$(4.10) \quad L^{\sigma e} - L^{e\sigma} = S^{\sigma\sigma} + 2 U^e \dot{I}^{\beta\sigma} U_\beta - 2 U^\sigma \dot{I}^{\beta e} U_\beta$$

and

$$(4.11) \quad M^{\mu\sigma\tau} - U^\tau L^{\mu\sigma} = \frac{1}{2} (U^\mu S^{\sigma\tau} - U^\sigma S^{\tau\mu} - U^\tau S^{\mu\sigma}) + \\ + \dot{I}^{\tau\mu\sigma} - \dot{I}^{\mu\sigma\tau} - \dot{I}^{\sigma\mu\tau} + 2 (U^\mu U^\sigma \dot{I}^{\beta\tau} U_\beta - U^\mu U^\tau \dot{I}^{\beta\sigma} U_\beta) .$$

It may readily be verified from eqs. (4.11) that eq. (2.8) is satisfied and that

$$(4.12) \quad \frac{1}{2} R_{\mu\sigma\tau}^e (M^{\mu\sigma\tau} - U^\tau L^{\mu\sigma}) = R_{\sigma\mu\tau}^e U^\mu S^{\sigma\tau} + R_{\mu\sigma\tau}^e \dot{I}^{\tau\mu\sigma} + 2 R_{\mu\sigma\tau}^e U^\mu U^\sigma \dot{I}^{\beta\tau} U_\beta$$

and hence

$$(4.13) \quad \frac{1}{2} U_e R_{\mu\sigma\tau}^e (M^{\mu\sigma\tau} - U^\tau L^{\mu\sigma}) = R_{\mu\sigma\tau}^e U_e \dot{I}^{\tau\mu\sigma} .$$

Equations (4.10), (4.12) and (4.13) will be needed below.

5. - Equations (2.9).

These equations perform two functions in the theory. First, their symmetric part defines the tensor $M^{e\sigma}$ in terms of the other tensors entering therein. Second, the antisymmetric part provides a differential equations for the determination of some of the components of $L^{\mu\nu}$ and the definition of the vector P^σ in terms of U^σ , $L^{\mu\nu}$ and $I^{\mu\nu}$. We shall discuss here only the antisymmetric part of eqs. (2.9). We find

$$(5.1) \quad P^e U^\sigma - P^\sigma U^e - \frac{D}{Ds} (L^{\sigma e} - L^{e\sigma}) + R_{\nu\beta\alpha}^e U^\nu U^\alpha \dot{I}^{\beta\sigma} - R_{\nu\beta\alpha}^\sigma U^\nu U^\alpha \dot{I}^{\beta e} = 0 .$$

Hence

$$(5.2) \quad P^e = m U^e + U_\sigma \frac{D}{Ds} (L^{\sigma e} - L^{e\sigma})$$

where as mentioned above

$$m = U_\sigma P^\sigma$$

and is to be interpreted as the mass of the quadrupole.

In view eqs. (4.10) we may write eqs. (5.2) as

$$(5.3) \quad P^e = m U^e + U_\sigma \frac{D}{Ds} S^{\sigma e} - 2 P_\sigma^e \frac{D}{Ds} [\dot{I}^{\beta\sigma} U_\beta] .$$

This equation reduces to the same one occurring in the theory of the dipole when $I^{\beta\sigma} = 0$.

On multiplying eqs. (5.1) by $P^\mu_\epsilon P^\nu_\sigma$ and summing we obtain

$$(5.4) \quad P^\mu_\epsilon P^\nu_\sigma \frac{D}{Ds} (L^{e\sigma} - L^{\sigma e}) = P^\mu_\epsilon R^e_{\gamma\beta\alpha} U^\gamma U^\alpha I^{\beta\nu} - P^\nu_\epsilon R^e_{\gamma\beta\alpha} U^\sigma U^\alpha I^{\beta\mu}$$

as three differential equations for the determination of the six components $S^{e\sigma}$ along the world line of the particle. The tensor $S^{e\sigma}$ must satisfy a supplementary condition of the type derived in ref. [1]. We shall discuss the derivation of this condition after we discuss some consequence of eqs. (2.10).

6. - Equations (2.10) and dm/ds .

In view of eqs. (3.4) and (4.12) these equations may be written as

$$(6.1) \quad \frac{DP^e}{Ds} + R^e_{\sigma\mu\tau} U^\mu S^{\sigma\tau} + R^e_{\mu\sigma\tau} \dot{I}^{\tau\mu\sigma} + 2R^e_{\mu\sigma\tau} U^\mu U^\sigma \dot{I}^{\tau\mu} U_\beta - U^\nu I^{\alpha\sigma} R^e_{\nu\tau\alpha;\sigma} = 0$$

On multiplying these equations by U_σ and summing we obtain

$$\frac{dm}{ds} - P^\sigma \frac{DU^\sigma}{Ds} + R^e_{\mu\sigma\tau} U_\epsilon \dot{I}^{\tau\mu\sigma} = 0$$

From eqs. (5.2) we see that

$$P_\epsilon \frac{DU^e}{Ds} = U_\sigma \frac{D}{Ds} (L^{e\sigma} - L^{\sigma e}) \frac{DU_\epsilon}{Ds} = \frac{D}{Ds} [U_\sigma (L^{e\sigma} - L^{\sigma e})] \frac{DU_\epsilon}{Ds}.$$

On making use of eqs. (4.10) we obtain

$$P_\epsilon \frac{DU^e}{Ds} = \frac{D}{Ds} [U_\sigma S^{e\sigma} - 2\dot{I}^{\beta e} U_\beta] \frac{DU_\epsilon}{Ds}$$

hence

$$(6.2) \quad \frac{dm}{ds} = \frac{Dv^e}{Ds} \frac{DU_\epsilon}{Ds} - R^e_{\mu\sigma\tau} U^e \dot{I}^{\tau\mu\sigma}$$

where

$$(6.3) \quad v^e = S^{e\sigma} U_\sigma - 2\dot{I}^{e\sigma} U_\sigma.$$

This vector will be discussed in the next Section. We note that there

are two reasons for the nonvanishing of dm/ds , the term involving v^e and the term involving the Riemann-Christoffel curvature tensor. Thus even when the underlying space is flat, that is, even when one considers a small perturbation from Minkowski space-time, the mass of the quadrupole need not be conserved along the world-line of the particle.

7. - The supplementary condition.

This Section will be devoted to the discussion of some consequences of eqs. (1.2) and (1.4). We first note that if we set $f_\mu = u_\mu f$ in the former equations it then becomes

$$U^e \int \rho f n_e dV = \mu f - \mu^\tau f_{;\tau} + \mu^{\sigma\tau} f_{;\sigma\tau}$$

where

$$(7.1) \quad \mu = N^\mu U_\mu - N^{\mu\sigma} U_{\mu;\sigma} + N^{\mu\sigma\tau} U_{\mu;\sigma\tau}$$

$$(7.2) \quad \mu^\tau = N^{\mu\tau} U_\mu - 2N^{\mu\sigma\tau} U_{\mu;\sigma}$$

$$(7.3) \quad \mu^{\sigma\tau} = N^{\mu\sigma\tau} U_\mu.$$

The condition that there be no mass dipoles may be expressed as

$$(7.4) \quad \mu^\tau = N^{\mu\tau} U_\mu - 2N^{\mu\sigma\tau} U_{\mu;\sigma} = 0.$$

We shall see that this condition together with eqs. (1.4) leads to a relation for v^e .

Multiplying eqs. (1.4) by f_μ , integrating and applying eqs. (1.1) and (1.2), we obtain after equating coefficients of derivatives of f_μ the following equations

$$(7.5) \quad (1 + \varepsilon/c^2) N^{\mu\sigma\tau} = M^{\mu\nu\sigma\tau} U_\nu$$

$$(7.6) \quad (1 + \varepsilon/c^2) N^{\mu\sigma} - 2N^{\mu\sigma\tau} (1 + \varepsilon/c^2)_{;\tau} = M^{\mu\nu\sigma} U_\nu - 2M^{\mu\nu\sigma\tau} U_{\nu;\tau}$$

$$(7.7) \quad (1 + \varepsilon/c^2) N^\mu - N^{\mu\sigma} (1 + \varepsilon/c^2)_{;\sigma} + N^{\mu\sigma\tau} (1 + \varepsilon/c^2)_{;\sigma\tau} = \\ = M^{\mu\nu} U_\nu - M^{\mu\nu\sigma} U_{\nu;\sigma} + M^{\mu\nu\sigma\tau} U_{\nu;\sigma\tau}.$$

It follows from eqs. (3.5), (4.6) and (6.3) that the first two of these equations may be written as

$$(7.8) \quad (1 + \varepsilon/c^2) N^{\mu\sigma\tau} = U^\mu I^{\sigma\tau}$$

and

$$(7.9) \quad (1 + \varepsilon/c^2)N^{\mu\sigma} = 2U^\mu I^{\sigma\tau} E_{,\tau} + \frac{1}{2}[S^{\mu\sigma} - U^\sigma v^\mu - U^\mu v^\sigma - 2\dot{I}^{\mu\sigma}]$$

where

$$E = \log(1 + \varepsilon/c^2).$$

Hence

$$(7.10) \quad N^{\mu\sigma} U_\sigma = 0$$

and

$$(1 + \varepsilon/c^2)U_\mu N^{\mu\sigma} = 2I^{\sigma\mu} E_{,\mu} - 2U_\mu \dot{I}^{\mu\sigma} - v^\sigma.$$

In view of eqs. (7.8), the condition (7.4) becomes

$$U_\mu N^{\mu\sigma} = 0$$

or

$$(7.11) \quad v^\sigma = 2I^{\sigma\mu} E_{,\mu} - 2U_\mu \dot{I}^{\sigma\mu} = 2I^{\sigma\mu}(E_{,\mu} + \dot{U}_\mu).$$

Equations (7.9) may be written as

$$(7.12) \quad (1 + \varepsilon/c^2)N^{\mu\sigma} = \frac{1}{2}(S^{\mu\sigma} + U^\mu v^\sigma - U^\sigma v^\mu - 2\dot{I}^{\mu\sigma}) + 2U^\mu U_\rho \dot{I}^{\rho\sigma}.$$

8. - The conservation of matter.

Equation (6.2) then becomes

$$(8.1) \quad \frac{dm}{ds} = 2 \frac{D}{Ds} \left(I^{\sigma\tau} \left(E_{,\tau} + \frac{DU_\tau}{Ds} \right) \right) \frac{DU_\sigma}{Ds} - R_{\rho\mu\sigma\tau} U^\rho \dot{I}^{\tau\mu\sigma}.$$

The change in m along the world line of the particle thus depends on $I^{\sigma\tau}$ and its derivative with respect to s as well as the derivative of E in the directions orthogonal to the world line. A partial determination of these latter quantities may be obtained from the discussion of the equation of the conservation of matter. As in ref. [1] we assume that the distribution ϱ satisfies a generalization of the equation

$$(\varrho u^\mu)_{;\mu} = 0.$$

We write this equation as

$$(\varrho u^\mu f)_{;\mu} = \varrho u^\mu f_{;\mu}$$

and integrate over a volume in four space to obtain

$$\int \varrho u^\mu f n_\mu dv = \int \varrho u^\mu f_{,\mu} \sqrt{-g} d^4x.$$

By using eq. (1.2) and the arguments of Sect. 2 we may obtain the equation

$$\frac{d}{ds} [wf - K^e f_{;e} + U_\alpha N^{\alpha e \sigma} f_{;e \sigma}] = N^e f_{;e} - N^{e \sigma} f_{;e \sigma} + N^{e \sigma \tau} f_{;e \sigma \tau}$$

where w and K^e are defined in terms of N^e , $N^{e \sigma}$ and $N^{e \sigma \tau}$ by equations analogous to eqs. (2.3) and (2.4).

Since this equation is to hold for arbitrary functions f we must have

$$(8.2) \quad P_\alpha^e (N^{\alpha \sigma \tau} + N^{\sigma \tau \alpha} + N^{\tau \alpha \sigma}) = 0$$

$$(8.3) \quad N^{e \mu} + N^{\mu e} = K^e U^\mu + K^\mu U^e - 2 \frac{D}{Ds} N^{\alpha e \mu} U_\alpha$$

$$(8.4) \quad w U^e - \frac{D}{Ds} K^e = N^e$$

$$(8.5) \quad \frac{dw}{ds} = 0.$$

These equations together with eqs. (7.5) through (7.7) provide additional relations for the twenty quantities m , U^σ , $S^{\sigma \tau}$ and $I^{\sigma \tau}$.

Equations (8.2) are satisfied as a consequence of eqs. (7.8) which in turn are equivalent to eqs. (7.5). Equations (8.3) provide a definition of K^e for it follows from them and (7.9) and (8.2) that

$$(8.6) \quad K_e U^e = 0$$

and

$$(8.7) \quad K^e = U_\alpha N^{\alpha e} + 2 U_\alpha \frac{D}{Ds} \left(\frac{I^{\alpha e}}{1 + \varepsilon/c^2} \right)$$

It follows from eqs. (8.3) and (7.12) that

$$(8.8) \quad (1 + \varepsilon/c^2) K^e = 2 U_\alpha \frac{D I^{\alpha e}}{Ds}$$

and that eqs. (8.3) are satisfied if and only if

$$I^{e \mu} \frac{dE}{ds} = 0.$$

In the case we are considering in which $I^{0\alpha} \neq 0$ this means that

$$\frac{dE}{ds} = 0.$$

That is, E is a constant along the world line. This does not imply that $\varepsilon_{,\alpha}$ or $\varepsilon_{,\sigma\tau}$ need be constants.

Equations (8.4) and (8.8) may be interpreted as differential equations for $I^{\mu\nu}$ along the world line of the particle if N^e is taken as given by eqs. (7.7). The latter equations do define N^μ but only in terms of the known tensor $M^{\mu\nu}$, $M^{\mu\nu\sigma}$, $M^{\mu\nu\sigma\tau}$ and the unknown quantities $E_{,\sigma}$, $E_{,\sigma\tau}$, $U_{,\sigma}$. Thus the theory given above does not lead to a closed set of equations for the quadrupole. In this respect this theory of quadrupole differs from that can be obtained for the dipole from the same set of assumptions. We have to supplement the above theory by some conditions which provide for the determination of $I^{\mu\nu}$. The Einstein field equations may do this.

Nevertheless it has been possible to show that the quantity m , which is reasonably interpreted as the mass of the quadrupole, is not conserved along the world line of the particle and its variation is given by eq. (8.1) in terms of $I^{\alpha\beta}$ which is related to the quadrupole moment of the particle.

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INTERVENTI E DISCUSSIONI

— W. B. BONNOR:

If the background field vanishes there is still a surviving term in (8.1). This seems to arise from internal motion of the particle. Is it a radiation phenomenon?

— A. H. TAUB:

The first term in eq. (8.1), the term which remains when the background field vanishes gives the interaction between the quadrupole moment, the derivatives of the internal energy in directions orthogonal to the motion and the derivatives of the velocity vector. The first of these quantities is the source of the perturbation in the gravitational field and may create « radiation », but since this perturbation does not occur explicitly in eq. (8.1) thus equation does not describe what is usually called a radiation phenomenon.

— H. BONDI:

The implications of these results become very clear when we consider the Earth-Moon system. The quadrupole part of the Moon's field raises tides, *i.e.* induces a quadrupole moment of the Earth. Due to the Earth's rotation and the action of dissipative forces (tidal friction), the Earth's tidal bulges are in advance of the Moon and so produce its secular acceleration, which of course immediately affects the motion of the Earth and so slows up the effect of the quadrupole moment on the orbit. Moreover, tidal friction heats the Earth and so increases its energy and hence its mass. We have therefore an inductive (not a radiative) transfer of energy through the gravitational field. Applying Taub's results to Møller's methods, we might test these in interesting circumstances.

The increase or decrease of mass clearly depends on the relation of the quadrupole field to the quadrupole moment. This may be induced, as in the Moon-Earth case. On the other hand, in the radiation case, we take the variation of quadrupole moment as given, calculate the field according to the boundary conditions assumed (*e.g.* outgoing waves only) and then we should be able to calculate the mass loss from Taub's equation.

— P. HAVAS:

As Dr. Taub already noted, his paper contains some results which appear to differ qualitatively from those obtained by several other authors even in the case of flat space, *i.e.* free particles. I believe that this is not necessarily the case, however.

As is well known, the equations of motion do *not* follow unambiguously from the field equations of general relativity if the particles are assumed to have a structure more complicated than simple poles of the gravitational field; if no conditions are imposed on the assumed structure, already in the case of a dipole particle the motion is not restricted at all. In the choice of the conditions to be imposed we can be guided by physical considerations concerning either the structure of the equations or the structure of the particles. From the first point of view we could simply demand that the mass be constant or, equivalently, that certain terms in the equations of motion should be interpretable as an intrinsic angular momentum. One way to achieve this in the case of flat space is to require the vanishing of the vector V^e (eqs. (6.3)). From the second point of view (discussed especially by Mathisson, 1937) we could demand that there should be no terms in the matter tensor which can be considered as the limit of an extended structure containing negative as well as positive masses. This implies not only the condition (7.4) on the mass dipole, but also conditions on the quadrupole, and it might well be the case that these would again lead to the constancy of the total mass. Other assumptions on the particle structure might lead to more complicated relations.

— A. H. TAUB:

I presently believe that eqs. (7.4) is the complete description of the requirement that the distribution ρ is the limit of an extended mass density containing only positive masses.

— A. PAPAPETROU:

In the frame of special relativity the problem of the equations of motion of particles with quadrupole structure has been discussed some time ago (PAPAPETROU: *Praktika Académie* (Athènes, 1944)). It turned then out that the motion of such a particle is much less determined by the equation of motion than the motion of a dipole particle. The reason is a simple one. The equations of motion are essentially the conservation laws of energy-momentum and of angular momentum plus the law of the mass-center and they therefore are just sufficient to determine the motion of a dipole particle, after restricting it by a subsidiary condition of the usual type (e.g. $\delta_{\mu\nu} u^\nu = 0$). The quadrupole particle has a number of new characteristic quantities, which remain unrestricted by the equations of motion.

It is not surprising that the same calculation can also be made in general relativity without encountering any fundamental difficulty when one considers the particle as a *test* particle (moving in a given gravitational field). It is to be stressed that these calculations constitute what one might conveniently call the zeroth order, as regards the contribution of the particle to the field. It would therefore be erroneous to expect any information about gravitational radiation emitted by the particle from the discussion of these equations of motion. In order to obtain such an information it will be indispensable to calculate (to some approximation) the contribution of the particle to the field.

The variability of the scalar m of the particle has nothing to do with a change of its energy. This follows at once from the remark that this variability exists also in special relativity, despite the fact that in this case the particle constitutes an isolated system and consequently its energy is conserved.

— A. H. TAUB:

There is a relation between the quantity m and the quantity $(1+E/c^2)$ which may be called the specific energy of the particle. This relation is provided by eq. (5.2), the symmetric part of eqs. (2.9) and the equation given in Sect. 3. I hope to discuss this relation in detail on another occasion.

— P. HAVAS:

Concerning Dr. Papapetrou's remarks on higher order approximations to the equations of motion of test particles, which might include radiation reaction terms, it was first pointed out by Schild, and later by other authors using different arguments, that such higher order terms are spurious.

Dr. Papapetrou also remarked that in special relativity there are ten conservation laws, which imply the ten translational and rotational equations of motion for a pole-dipole particle, but that quadrupoles and higher poles are not restricted by further conservation laws.

However, it was shown by Shanmugadhasan about twenty years ago using Mathisson's method that no matter how high the multipole moment involved, the translational and rotational equations of motion are always of the same form as for a pole-dipole particle; it is only the physically irrelevant definition of the momentum and spin of the particle in terms of the moments which is

different. There do not exist any further equations of motion due to the multipoles.

— H. BONDI:

Equations of motion are not an approximation of some order to the field equations, but separate from them. The field in Taub's work may be calculated to the zeroth or higher order and then corresponding results will follow.

As for «mass», this is really only defined for a static monopole (or similar system) in asymptotically flat space. If therefore Taub's particle starts and finishes as a monopole in asymptotically flat space, then the *charge* in mass has a clear meaning.

Thus if dm/ds is always of one sign, then the significance of these changes is immediately clear.

— W. D. BEIGLBÖCK:

I've a question to the interpretation: when describing an extended body by multipoles we in general need a countable number of multipole-moments. On the other hand, when starting with Schwartz-distributions, there exists only a *finite* number of multipole-moments because of a classical theorem of Schwartz. Therefore one cannot expect a one-to-one correspondence between these both sorts of multipole-moments. Does this one-to-one correspondence hold at least for the first multipole-moments? Were you dealing with?

— A. H. TAUB:

If the tensor $f_{\mu\nu}$ entering into eq. (1.1) is written as:

$$f_{\mu\nu} = a_{\mu\nu} + b_{\mu\nu\sigma}(x^\sigma - X^\sigma) + c_{\mu\nu\sigma\tau}(x^\sigma - X^\sigma)(x^\tau - X^\tau) + \dots$$

with $a_{\mu\nu}$, $b_{\mu\nu\sigma}$ and $c_{\mu\nu\sigma\tau}$ constant, we may obtain relations between the $M^{\mu\nu}$, $M^{\mu\nu\sigma}$ and $M^{\mu\nu\sigma\tau}$ and the usually defined multipole moments. The finite number of multipole moments entering in the use of Schwartz distributions is a restriction but I do not believe this to be a serious one.

— J. A. WHEELER:

The work of Taub, Papapetrou and others is helping step by step to elucidate important issues concerned with the *meaning* of a *limitations* on the concept of «equations of motion».

1) One sees the principle of action and reaction at work. The action of the moving object upon its surroundings is described by its moments. The action of the background metric upon the object is described by such derivatives of the metric as appear in the Riemann curvature tensor and in higher order curvature tensors. The coupling of the moments of various orders with the curvatures of various orders gives a simple system of equations for the center of gravity motion and for the time rate of change of the angular momentum but *not* for the quadrupole and higher moments, because the time rate of change of these higher moments depend upon the *details* of the dynamics internal to the object (Table).

TABLE. — *The principle of action and reaction predicts a different role (in the equations of motion) for quadrupole and higher moments than for angular momentum and center of mass.*

Moment of the object	Effect of object on surrounding field when geometry is asymptotically flat	Effect of background field (when not flat) upon the moments of the object
Energy-momentum	Constant (general theorem)	Motion of center-of-mass shown to follow geodesic equation of motion (lowest order analysis, with higher order corrections associated with effect of higher moments) by use of Einstein field equations solely in region <i>surrounding</i> the object
Angular momentum	Constant (general theorem)	Time rate of change of angular momentum governed by appropriate derivatives of background metric; this proven by use of Einstein field equations solely in regions <i>surrounding</i> the object
Quadrupole and higher moments	Not in general constant; governed by dynamics interior to object (astronomical object or geon)	Time rate of change of quadrupole moment governed by dynamics internal to system; <i>cannot</i> be predicted from Einstein-Infeld-Hoffmann analysis or any other analysis which uses field equations solely in region <i>exterior</i> to the object

The structure of the equations of motion found by Professor Taub, as also those found earlier by Professor Papapetrou, is consistent with this picture: one can find the motion of the center-of-mass, and the time rate of change of the angular momentum from the application of Einstein's field equations solely in the region outside the object; but one cannot find in this way the time rate of change of the higher moments.

2) One has in the tidal oscillation of a planet a well known *example* where are kinds of details (such as the location of the Bay of Fundy!) affect the time rate of the quadrupole moment under the tide producing action of the background metric.

3) The work done on the Earth by the tidal forces is converted into heat; heat appears as energy; and energy has mass. Therefore the mass of the system, quite naturally, changes at a rate governed by the time rate of change of the quadrupole moment (*internal* dynamics) as seen in the equations of Taub and Papapetrou, and as also elucidated on physical grounds by Bondi.

4) Another model for an object is a geon. A simple toroidal geon is endowed with a quadrupole moment. Moreover, the quadrupole moment of the general geon, like the quadrupole moment of the Earth, changes with time in a complicated way, depending upon internal dynamics, whether or not there is any external tide producing force.

5) These examples show that one will *never* be able to derive the equations of motion in all detail solely by following the program of Einstein-Infeld-Hoffmann and limiting attention to the region *exterior* to the object.

6) Moreover, one can even see from these models that there is a certain sense in which *the concept of «equation of motion» completely breaks down*. One is familiar in tidal theory with the Roché limit. This limit measures the critical tide producing force which is required to break up a planet into fragments. A similar fission of a geon into parts will also take place when the tide producing force is sufficiently large, according to a simple order of magnitude analysis (J. A. WHEELER: *Geometrodynamics* (New York, 1962)). Therefore «one» equation of motion must explode into a number of equations of motion. It would be completely out of place to try to treat the ever wider separating fragments by the single equation of motion of the original planet! To be sure, there is a certain sense in which this separation of the fragments can be called «internal dynamics». However, to treat this separation as «internal dynamics» is more and more inappropriate as the distances become larger and larger.

7) What consequence does explosion have for the structure of the «equations of motion» with their infinity of terms? The existence of fragmentation processes (Roché limit) tells one that *the series cannot make sense* when pursued term after term to terms of indefinitely high order. From this circumstance it is natural to expect that the series in question must be *divergent*.

8) Despite divergence the first terms of the series (center-of-mass motion; time rate of change of angular momentum, quadrupole moment, and other low order moments) obviously make good sense under most physical circumstances. In this respect it would appear that there is a certain analogy (cf. J. A. WHEELER: *Geometrodynamics and the Problem of Motion*, *Wigner issue of Reviews of Modern Physics* (1963)) between the power series for the motion and the power series for the energy level in a well known problem of quantum mechanics. Let a particle move in the potential $V(x) = x^2 - \lambda x^3$, with λ small. Then the n -th energy level E_n differs very little from that of a harmonic oscillator, if one judges from the first terms in the power series for $E_n(\lambda)$ (BORN: *Elementare Quantenmechanik*). However, the series diverges! The reason is simple. There is no such thing (in principle) as an energy level any more after the x^3 term is introduced in the potential.

There exists a barrier of finite height through which a particle can always tunnel and escape. Thus there is a good reason why the series should diverge: it describes something (an energy level) which has no physical existence! Similarly in the series for the motion of an object—which in principle always can and eventually will fragment. Yet in both cases the divergence of the series is in most practical cases of no concern, because the first few terms converge so rapidly.

In conclusion, the work of Professor Taub reminds us again of the sophisticated sense in which the very idea of any equation of motion at the same time does, and does not, make sense.

— O. COSTA DE BEAUREGARD:

Je m'intéresse moi aussi à la théorie du spin depuis longtemps (1942) mais pas du même point de vue que le Professeur Taub. Je m'intéresse à la théorie des milieux continus, et notamment à la théorie du fluide statistique associé à l'électron de Dirac; ceci dans l'espace-temps plat.

Je recherche depuis longtemps si, physiquement parlant, il y a ou non un sens à dire que, localement, la densité d'impulsion et la densité de courant de Dirac son non colinéaires.

Finalement j'ai proposé une expérience basée sur le ferromagnétisme (qui est du au spin de l'électron). L'expérience, fort délicate, a pu être effectuée récemment par mon collaborateur Ch. Goillot, et elle a donné un résultat parfaitement conforme (y compris l'ordre de grandeur numérique) avec la théorie que j'avais proposée.

La conclusion est donc que, dans le fluide à spin associé à l'électron de Dirac, la vitesse et l'impulsion sont *non colinéaires*.

Un point digne de remarque est que le résultat positif a été obtenu avec un corps d'épreuve ferromagnétique (alliage Fe-Ni-Co) où le magnétisme est dû en partie aux électrons de conduction. Une expérience préalable moins précise faite avec des ferrites nous avait donné un résultat nul: il apparaît ainsi que le caractère *fluide* du contenu du corps d'épreuve doit être essentiel à l'obtention d'un résultat non nul. Ce point important m'avait échappé lors de mes publications initiales sur ce sujet; c'est à la suite de conversations privées avec MM. Papapetrou, Bell et d'Espagnat que je l'ai pris en considération.

[Alla discussione partecipò anche A. SCHILD.]

Approximate Methods and Gravitational Radiation.

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1. - Introduction.

In recent years many new exact solutions of Einstein's field equations have been discovered. Some of these, notably that of Bondi, Pirani and Robinson (1959), have made it seem likely that, according to general relativity, gravitational waves have a genuine existence.

Nevertheless, to deal with realistic physical situations, it is necessary to use approximations. In this paper I shall review the three principal approximation methods that have been used to study gravitational radiation from finite, isolated sources. First, there is the method of Einstein, Infeld and Hoffmann (EIH); secondly, the fast approximation method; and thirdly, Bondi's method. The first has been applied only to systems of freely gravitating particles. In the last two the nature of the source need not be specified (except possibly in regard to symmetry and finiteness) and non gravitational forces may be present.

The three methods have one thing in common. They all depend on solutions of the inhomogeneous wave equation,

$$(1.1) \quad \square \varphi \stackrel{\text{def}}{=} \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = f(x, y, z, t),$$

where the source-function f is a delta function. The differences in the solutions chosen account to a certain extent for the differences in the methods.

What workers in this field are trying to do is this. From the linear approximation together with the energy pseudo tensor there follows an unambiguous prediction that outgoing waves from isolated sources carry energy. One therefore naturally wants to find out what happens to the source during the radiation process; for example, does it lose an amount of energy equal to that allegedly carried off by the waves? The linear

approximation cannot tell one this, so it is necessary to go to higher approximations.

Before describing the methods I give briefly in Sect. 2 the standard theory of the linear approximation which is, however, so well known that there is no need to dwell upon it.

Concerning notation, I shall take for the background Minkowski space-time

$$(1.2) \quad ds^2 = -(\mathbf{d}x_1^2 + \mathbf{d}x_2^2 + \mathbf{d}x_3^2) + \mathbf{d}x_4^2 \stackrel{\text{def}}{=} \eta_{ik} \mathbf{d}x^i \mathbf{d}x^k,$$

this being the definition of η_{ik} ; the η^{ik} are defined by

$$\eta^{ik} \eta_{ij} = \delta_j^k.$$

The field equations are

$$(1.3) \quad G_{ik} \stackrel{\text{def}}{=} R_{ik} - \frac{1}{2} g_{ik} R = -8\pi k T_{ik},$$

where k is the gravitational constant. Latin indices run from 1 to 4 and greek from 1 to 3. A comma will mean partial differentiation, for example

$$g_{ik,l} = \frac{\partial g_{ik}}{\partial x_l}.$$

Except in Sect. 3 I shall choose units so that $c = k = 1$.

2. - The linear approximation.

In this Section I assume that the gravitational field is weak and write

$$(2.1) \quad g_{ik} = \eta_{ik} + \gamma_{ik};$$

products of γ_{ik} and of their derivatives are to be neglected. If we now introduce the auxiliary quantities

$$(2.2) \quad \gamma_{ik}^* = \gamma_{ik} - \frac{1}{2} \eta_{ik} \eta^{ab} \gamma_{ab},$$

and choose the co-ordinates so that γ_{ik}^* satisfies the harmonic co-ordinate condition

$$(2.3) \quad \eta^{kl} \gamma_{ik,l}^* = 0$$

the field eqs. (1.3) become

$$(2.4) \quad \eta^{ab} \gamma_{ik,ab}^* = -16\pi T_{ik}.$$

The inhomogeneous wave eqs. (2.4) have wave solutions of the expected types. The equations and some solutions of them were given in 1916 by Einstein, and for twenty years thereafter the theory of gravitational radiation was based on them. Among other early results of this theory was that, according to the energy pseudo tensor, there is a transmission of energy by the waves from a bounded source. Quite recently it has been shown, once again by the use of the pseudo tensor, that momentum too is transported if the source vibrates in a suitable way (Bonnor and Rotenberg, 1961; Papapetrou, 1962; Peres, 1962).

3. - The EIH method.

This method originated in a famous paper by Einstein, Infeld and Hoffmann (1938). It has been refined since, especially by Infeld, and the results of twenty years work on it are given in the book *Motion and Relativity* (Infeld and Plebanski, 1960). We are here interested only in the results given by the method for gravitational waves emitted by freely gravitating particles. I adopt the point of view of Trautman (1958).

Take the scalar wave eq. (1.1) with $f=0$, and write it as

$$(3.1) \quad \square\varphi - \frac{1}{c^2} \ddot{\varphi} = 0$$

where ∇ is the Laplacian operator and the dot means differentiation with respect to the time. The starting point of the EIH approximation method is to consider solutions that can be expanded in a series of ascending powers of c^{-1} :

$$(3.2) \quad \varphi = \sum_{n=0}^{\infty} c^{-n} \varphi(t, x_{\alpha})_{(n)}.$$

Substituting (3.2) into (3.1) and equating to zero successive coefficients of the series in c^{-n} , we obtain

$$(3.3) \quad \nabla\varphi_{(0)} = 0,$$

$$(3.4) \quad \nabla\varphi_{(1)} = 0,$$

$$(3.5) \quad \nabla\varphi_{(2)} = \ddot{\varphi}_{(0)},$$

.

$$(3.6) \quad \nabla\varphi_{(k)} = \ddot{\varphi}_{(k-2)}, \quad \text{for } k \geq 2.$$

If we wish, we can take only the even terms in φ , or only the odd ones. Let us put

$$(3.7) \quad \underset{(0)}{\varphi} = 0, \quad \underset{(2n-1)}{\varphi} = 0 \quad \text{for } n > 0,$$

so that the first of (3.3)-(3.6) we have to solve is

$$(3.8) \quad \underset{(2)}{\nabla \varphi} = 0.$$

We now take the monopole solution of (3.8)

$$(3.9) \quad \underset{(2)}{\varphi} = r^{-1}\alpha(t);$$

this really requires us to reinstate f as a delta function on the right hand side of (3.1) but for simplicity we ignore this. As a solution of the eqs. (3.6) subject to (3.7) we may take

$$(3.10) \quad \underset{(2)}{\varphi} = r^{-1}\alpha(t), \quad \underset{(4)}{\varphi} = \frac{1}{2} r \ddot{\alpha}, \quad \underset{(6)}{\varphi} = \frac{1}{4!} r^3 \overset{..}{\ddot{\alpha}},$$

although this is, of course, not unique (*). φ given by (3.10) is a standing wave solution of (3.1), namely,

$$(3.11) \quad \varphi = (2rc^2)^{-1} \left\{ \alpha \left(t - \frac{r}{c} \right) + \alpha \left(t + \frac{r}{c} \right) \right\}.$$

A retarded solution of (3.1),

$$(3.12) \quad \varphi = (rc^2)^{-1} \alpha \left(t - \frac{r}{c} \right)$$

is obtained by choosing the following set of solutions of (3.3)-(3.6):

$$(3.13) \quad \begin{cases} \underset{(0)}{\varphi} = \underset{(1)}{\varphi} = 0, & \underset{(2)}{\varphi} = r^{-1}\alpha, & \underset{(3)}{\varphi} = -\dot{\alpha}, \\ \underset{(4)}{\varphi} = \frac{1}{2} r \ddot{\alpha}, & \underset{(5)}{\varphi} = \frac{1}{3!} r^2 \overset{..}{\ddot{\alpha}}, & \dots \end{cases}$$

(*) Why we start at $\underset{(2)}{\varphi}$ rather than $\underset{(0)}{\varphi}$ will become clear later in this Section.

It should be noted that for $n \geq 4$ successive terms in the series (3.2) tend to infinity with r (which is not to say that the series diverges), so that *the expansion is not suitable for studying the field far from the source*. This remark applies to the whole of the EIH method, and means that one cannot, for example, identify the Schwarzschild mass of the system.

To see for what values of r the higher terms do become small, we imagine $\alpha(t)$ to be some oscillatory function, *e.g.*

$$\alpha = a \sin(ct/\lambda),$$

where λ can be taken as a characteristic wave length of the system. Then for $n \geq 2$,

$$c^{-n} \varphi_{(n)} \sim c^{-n} r^{n-3} \frac{d^{n-2} \alpha}{dt^{n-2}},$$

we have

$$c^{-n} \varphi_{(n)} \sim \frac{a}{rc^2} \left(\frac{r}{\lambda} \right)^{n-2},$$

and this will be small if $\lambda \gg r$ and $r \sim a$. Thus, the EIH method is not suitable for the description of the wave zone.

The EIH method applied to the gravitational field consists in expanding the field eqs. (1.3) in ascending powers of c^{-1} . To show how this is done I shall use dimensional units. Let us take the Minkowski metric in the form

$$(3.14) \quad ds^2 = -c^2(d\bar{x}_1^2 + d\bar{x}_2^2 + d\bar{x}_3^2) + d\bar{x}_4^2,$$

and for the time being confine ourselves to the linearized theory but use the EIH method. Write

$$(3.15) \quad \begin{cases} g_{\alpha\beta} = -\frac{1}{c^2} + \frac{\varepsilon_{\alpha\beta}}{c^2}, \\ g_{\alpha 4} = \frac{\varepsilon_{\alpha 4}}{c}, \\ g_{44} = 1 + \varepsilon_{44}, \end{cases}$$

where the products of ε_{ik} and of their derivatives are to be neglected. We then find, proceeding as in Sect. 2, that if we define

$$(3.16) \quad \varepsilon_{ik}^* = \varepsilon_{ik} - \frac{1}{2} \eta_{ik} \eta^{ab} \varepsilon_{ab}$$

(η_{ik} being as defined in (1.2)) then the co-ordinates may be so chosen

that the linearised field equations become

$$(3.17) \quad \begin{cases} \square \varepsilon_{\alpha\beta}^* = 16\pi k T_{\alpha\beta} , \\ \square \varepsilon_{4\alpha}^* = 16\pi k c^{-1} T_{4\alpha} , \\ \square \varepsilon_{44}^* = 16\pi k c^{-2} T_{44} , \end{cases}$$

$$(3.18) \quad \eta^{\alpha\beta} \varepsilon_{i\alpha,\beta}^* + c^{-1} \varepsilon_{i4,4}^* = 0 ,$$

where differentiations are carried out with respect to the co-ordinates $\bar{x}_i$ of (3.14).

We now examine the right hand side of (3.17). Let us for the moment neglect gravitation and think of matter in the form of dust at rest with rest density ϱ , so that its energy tensor is $T^{ik} = \varrho \delta_4^i \delta_4^k$. If we carry out a Lorentz transformation to a frame moving along the axis of $\bar{x}_3$ with speed v , we find for T_{ik}^* in the moving frame, provided $v \ll c$,

$$T_{33}^* \sim \frac{\varrho v^2}{c^4} , \quad T_{34}^* \sim \frac{\varrho v}{c^2} , \quad T_{44}^* \sim \varrho .$$

We accordingly suppose that for matter moving slowly with respect to the frame of (3.14) the leading terms in T_{ik} are

$$(3.19) \quad T_{\alpha\beta} \sim c^{-4} , \quad T_{\alpha 4} \sim c^{-2} , \quad T_{44} \sim c^0 .$$

From (3.17) we shall therefore have

$$(3.20) \quad \varepsilon_{\alpha\beta}^* \sim c^{-4} , \quad \varepsilon_{\alpha 4}^* \sim c^{-3} , \quad \varepsilon_{44}^* \sim c^{-2} .$$

We now revert from the linearized theory to general relativity with nonlinear terms. We assume that the g_{ik} may be expanded in powers of c^{-1} , adopt once again (3.15), but no longer neglect products of ε_{ik} . In fact we expand ε_{ik} themselves in powers of c^{-1} , which means because of (3.16) that the ε_{ik}^* are also expanded in this way:

$$\varepsilon_{ik}^* = \sum_{(n)} c^{-n} \varepsilon_{ik}^{*(n)} .$$

Expanding the field eqs. (1.3) in powers of c^{-1} and equating the coefficients to zero (*) we find, having regard to (3.19) and (3.20) that

(*) Strictly this procedure needs further justification because the coefficients themselves depend on c . They involve c through the co-ordinates of the particles, which depend on the equations of motion in which c enters. This was first pointed out by Narlikar and Rao (1955) who showed that nevertheless EIH had obtained correct equations of motion up to the post-Newtonian approximation. See also Goldberg (1962) and Sect. 4 of this paper.

(3.17) is changed to

$$(3.21) \quad \begin{cases} \nabla_{(2)} \varepsilon_{44}^* = 16\pi k T_{(0)44}, \\ \nabla_{(8)} \varepsilon_{4\alpha}^* = 16\pi k T_{(2)4\alpha}, \\ \nabla_{(4)} \varepsilon_{,\beta}^* = 16\pi k T_{(4)\alpha\beta} + N_{(4)\alpha\beta}, \end{cases}$$

where $N_{(4)\alpha\beta}$ are nonlinear terms entering for the first time. The co-ordinate condition (3.18) becomes

$$(3.22) \quad \begin{cases} \eta_{(3)}^{\alpha\beta} \varepsilon_{4,\beta}^* + \varepsilon_{(2)44}^* = 0, \\ \eta_{(4)}^{\alpha\beta} \varepsilon_{\mu\alpha,\beta}^* + \varepsilon_{(3)\mu 4}^* = 0. \end{cases}$$

The EIH method in general relativity consists in solving (3.21) and (3.22) (and the corresponding higher equations) near the moving particles (*). From the solutions, the equations of motion of the particles may be obtained by a procedure with surface integrals which does not concern us here. The sources are all point singularities, and the problem considered is that of freely gravitating particles (**). In the earlier papers T_{ik} was put zero but in the later work of Infeld and others it was, more correctly, expressed in terms of δ -functions. In either case, it is an essential feature of the method that the solution of the lowest equation (the first of (3.21)) shall represent a number of spherically symmetric mass points (**):

$$(3.23) \quad \varepsilon_{(2)44}^* = - \sum_{(s)} \frac{4km^{(s)}}{|x^\alpha - y^\alpha|^{(s)}},$$

where $m^{(s)}, y^\alpha$ are the mass and co-ordinates of the s -th particle and the summation is over all the particles.

(*) In obtaining (3.21) we used the co-ordinate condition (3.22). The development of the method does not depend essentially on this condition. See Einstein and Infeld (1940, 1949).

(**) For an extension of the EIH method to cases in which non gravitational forces are present, see Bonnor (1959b).

(**) See, however, Kerr (1959).

To solve the problem of motion one *can* expand the ε_{ik}^* as follows:

$$(3.24) \quad \begin{cases} \varepsilon_{\alpha\beta}^* = \sum_{n=1}^{\infty} c^{-(2n+2)} \varepsilon_{\alpha\beta}^*{}_{(2n+2)}, \\ \varepsilon_{4\alpha}^* = \sum_{n=1}^{\infty} c^{-(2n+1)} \varepsilon_{4\alpha}^*{}_{(2n+1)}, \\ \varepsilon_{44}^* = \sum_{n=1}^{\infty} c^{-2n} \varepsilon_{44}^*{}_{(2n)}. \end{cases}$$

Because of the similarity of (3.10) and (3.24) the latter is taken to refer to «standing wave» solutions of Einstein's equations. These are non-radiative and include steady-state solutions for orbiting particles.

Solutions representing radiating particles arise—if indeed they arise at all—when extra terms are inserted in (3.24) corresponding to the extra terms in (3.13). The first candidates for the extra terms are $\varepsilon_{44}^*{}_{(3)}$ and $\varepsilon_{4\alpha}^*{}_{(4)}$, and the equations to be satisfied are

$$(3.25) \quad \begin{cases} \nabla \varepsilon_{44}^*{}_{(3)} = 0, & \nabla \varepsilon_{4\alpha}^*{}_{(4)} = 0, \\ \gamma^{\alpha\beta} \varepsilon_{4\alpha,\beta}^*{}_{(4)} + \varepsilon_{44,4}^*{}_{(3)} = 0. \end{cases}$$

It is at this stage that the arbitrariness of the EIH method when applied to radiation becomes apparent. There are evidently a great many possible solutions of (3.25) which might be used. It has been the practice to follow the electrodynamic solution as closely as possible, and this has led workers to take

$$(3.26) \quad \begin{cases} \varepsilon_{44}^*{}_{(3)} = 0, \\ \varepsilon_{4\alpha}^*{}_{(4)} = -4 \sum_{(s)} m \frac{d^2 y^\alpha}{d\bar{x}_4^2}. \end{cases}$$

Infeld and Scheidegger (1951) showed that the terms in (3.26) can be removed by a co-ordinate transformation. Indeed Infeld seems to be of the opinion that the radiation terms can *always* be annihilated. However, other authors mostly agree that genuine radiation terms can be inserted higher up the series.

Trautman (1958), following a suggestion of Goldberg (1955), takes as the first radiative pair $\varepsilon_{\alpha\beta}^*{}_{(5)}$, $\varepsilon_{4\alpha}^*{}_{(6)}$. He obtains $\varepsilon_{44}^*{}_{(7)}$ by solving the field equations and applies the result to the double star problem, finding that there is a secular change in the motion due to the radiation terms. He remarks, however, that starting from his $\varepsilon_{\alpha\beta}^*{}_{(5)}$ it is possible to obtain either

the «genuine» radiation field, or a «flat» radiation field that can be transformed away. This is another indication of the nonuniqueness of the method. In the «genuine» case the damping term in the equation of motion is of order c^{-6} . However, Trautman points out that the physical meaning of the equation is obscure: for example, one cannot say whether the physical distance of the stars is increasing or decreasing.

Several other workers have used the EIH method to investigate radiation emitted by gravitating particles. The first study of the double star problem was that of Hu (1947), who found that the system *gains* energy whilst radiating. A similar result was found by Peres (1959), but he subsequently retracted it in favour of an energy loss (1960). Ryteń (1963) finds positive damping on account of radiation.

4. - The fast approximation method.

In this method one expands the g_{ik} , not in powers of c^{-1} but in powers of k , the gravitational constant, or some equivalent parameter. Among those who have used the method are Bertotti (1956); Kerr (1959), Havas and Goldberg (1962) and other authors cited below.

Let us suppose that an exact solution of the field eqs. (1.3) consists of g_{ik} , T_{ik} which are functions of a parameter λ and analytic in λ in the neighbourhood of $\lambda = 0$. Then the solution can be expanded in integral powers of λ :

$$(4.1) \quad \begin{cases} g_{ik}(x_a, \lambda) = \sum_{n=0}^{\infty} \lambda^n g_{ik}^{(n)}(x_a), \\ T_{ik}(x_a, \lambda) = \sum_{n=0}^{\infty} \lambda^n T_{ik}^{(n)}(x_a), \end{cases}$$

where the coefficients $g_{ik}^{(n)}$ and $T_{ik}^{(n)}$ are independent of λ . It will then follow that if we substitute (4.1) into the field equations written in the form (k having been absorbed into T_{ik})

$$(4.2) \quad G_{ik} + 8\pi T_{ik} = 0$$

we may equate to zero all coefficients of λ^n . The mathematical justification for this is that the function on the right of (4.2), namely zero, has a Taylor expansion in λ with zero coefficients: so, therefore, has the left hand side. We thus obtain

$$(4.3) \quad G_{ik}^{(n)} + 8\pi T_{ik}^{(n)} = 0, \quad n = 0, 1, \dots$$

As pointed out in a footnote in Sect. 3 this situation does *not* apply to the ordinary EIH method because the g_{ik} chosen, *e.g.* those obtained from (3.23), are not independent of the $\gamma_{ik}^{(n)}$ parameter. Nor does it apply to the fast approximation method when it is used to obtain equations of motion of particles. These methods are really procedures of iteration rather than genuine power series expansions. However, the previous paragraph does apply to the fast method as used by Fock and by Bonnor (see below).

If we let $\lambda = k$ we can separate the field equations (now written in form (1.3), not (4.2)) into approximation steps:

$$(4.4) \quad G_{ik}^{(n)} + 8\pi T_{ik}^{(n-1)} = 0, \quad n = 1, 2, \dots$$

The first approximation:

$$(4.5) \quad G_{ik}^{(1)} + 8\pi T_{ik}^{(0)} = 0$$

gives the linearized theory of Sect. 2. We can generalize this theory by writing

$$(4.6) \quad g_{ik} = \eta_{ik} + \sum_{n=1}^{\infty} k^n \gamma_{ik}^{(n)},$$

introducing the usual auxiliary quantities

$$(4.7) \quad \gamma_{ik}^{*} = \gamma_{ik}^{(n)} - \frac{1}{2} \eta_{ik} \eta^{ab} \gamma_{ab}^{(n)},$$

and the co-ordinate condition

$$(4.8) \quad \eta^{ab} \gamma_{ia,b}^{*} = 0.$$

Then (4.4) becomes for $n \geq 2$

$$(4.9) \quad \eta^{ab} \gamma_{ik,ab}^{*} = -16\pi T_{ik}^{(n-1)} - 2N_{ik}^{(n)},$$

where $N_{ik}^{(n)}$ are nonlinear terms known from previous approximations (they contain no terms in γ_{ik}^{*}).

This applies only to the case where the $\gamma_{ik}^{(n)}$ are independent of k . However, the formalism will also be useful later when we describe work on gravitational motion of particles. For the time being we describe nongravitational motion, in which the $\gamma_{ik}^{(n)}$ are independent of k . We

consider an isolated system of masses made to move in some arbitrary way by machinery, and confine ourselves to the first and second approximations. The equations to be solved are

$$(4.10) \quad \eta^{ab} \gamma_{ik,ab}^{*} = -16\pi T_{ik}^{(0)},$$

$$(4.11) \quad \eta^{ab} \gamma_{i,ab}^{*} = 0,$$

$$(4.12) \quad \eta^{ab} \gamma_{ik,ab}^{*} = -16\pi T_{ik}^{(1)} - 2N_{ik}^{(2)},$$

$$(4.13) \quad \eta^{ab} \gamma_{i,ab}^{*} = 0.$$

As source for (4.10) one might take, for instance, a quadrupole oscillator, obtaining a γ_{ik}^{*} corresponding to a quadrupole field. The question arises what does one insert for the $T_{ik}^{(1)}$ in (4.12)? The answer seems to be that one should allow only such sources as are necessary to prevent singularities which extend to infinity in the γ_{ik}^{*} . To make this clearer let us recall that a correction of order k to the mass of the source (roughly to $\int T_{44}^{(0)} dv$) is to be expected if the gravitational waves remove energy at a rate linearly dependent on k . This correction should turn up in $T_{ik}^{(1)}$. If we do not allow it, we shall need singularities extending to infinity representing, for example, pipes through which additional matter can be fed to make up the loss.

Actually, the solution of (4.10)-(4.13) has not been attempted in quite this way because of the difficulty of setting up energy tensors for these systems (*). Fock (1957, 1959) has looked for a solution with $T_{ik}^{(0)}$ and $T_{ik}^{(1)}$ zero, but allowing singularities at the origin of co-ordinates. This amounts to allowing the sources to be represented by an unspecified delta function. Fock examined the field in the wave zone ($r \gg \lambda$), and found that the γ_{ik}^{*} show secular changes which he interprets as being due to transmission of energy from the source by the waves. Unfortunately, his metric does not tend to Schwarzschild values at great distance even after the waves have passed by, because there remain terms of order $(\log r)/r$. It seems that the harmonic co-ordinate system is not suitable for dealing with this problem.

Later work has shown that for success in this problem good fortune in the choice of a co-ordinate system is most desirable. In my own work (Bonnor, 1959a) I considered the distant field of a very simple source

(*) Rosen and Shamir (1957) succeeded in doing this for a special system in the linear approximation, through not in harmonic co-ordinates.

consisting of two equal masses vibrating symmetrically at the ends of a spring. For this axially symmetric system it is not desirable to use harmonic co-ordinates. One can use pseudo spherical polar co-ordinates and a diagonal metric (*)

$$(4.14) \quad ds^2 = -A dr^2 - Br^2 d\theta^2 - Cr^2 \sin^2 \theta d\varphi^2 + D dt^2,$$

where A , B , C and D are functions of r , θ and t .

The method of approximation is somewhat different from the usual fast approximation expansion, and arose from the following considerations. From the linear approximation we know that the wave field for this system will depend on the multipole moments. These involve two parameters, m , the value of one of the masses, and a , a characteristic length in terms of which the displacements of the masses from their centre of mass are

$$(4.15) \quad \zeta_1(a, t) = -\zeta_2(a, t) = af(t),$$

where $f(t)$ is independent of m and a . Let us suppose that the field equations admit solutions that can be expanded in convergent Taylor series in m and a about $m=0$ and $a=0$ (**):

$$(4.16) \quad g_{ik} = \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} m^p a^s g_{ik}^{(ps)},$$

where $g_{ik}^{(ps)}$ are independent of m and a . The ordinary expansion in powers of k is equivalent to an expansion in powers of m only.

I use the field eqs. (1.3) with $T_{ik}=0$ except at the origin where it is given by an unspecified delta function, as in Fock's work. This means that I consider only the field in empty space outside the source.

If we substitute (4.16) into these field equations and equate to zero the coefficient of $m^p a^s$ we get what I call the (ps) approximation. The (00) and (10) approximations refer to flat space-time and to the linearized Schwarzschild solutions of mass $2m$, respectively. The higher $(1s)$ approximations are all linear—that is, each set of $g_{ik}^{(1s)}$ satisfy a set of linear equations—and they correspond to the 2^s -pole oscillation of the system.

It is clear that this method of approximation analyses the field into its component multipole fields. In fact, one of the $g_{ik}^{(1s)}$ —namely $g_{11}^{(1s)}$ —

(*) Though, as will be explained in Sect. 5, great simplifications arise if one chooses a certain non-diagonal metric.

(**) The notation here is slightly different from that of Bonnor (1959a) in which the exponent of a was $2s$.

satisfies the wave equation in spherical polar co-ordinates, and one chooses the solution corresponding to the 2^s -pole. For instance, the quadrupole term is

$$(4.17) \quad ma^2 g_{11}^{(12)} = 2r^{-1} \cos^2 \theta \ddot{Q} + 4P_2(r^{-2} \dot{Q} + r^{-3} Q),$$

where P_2 is the Legendre polynomial, $Q(t-r)$ is the quadrupole moment with $t-r$ in place of t as argument, and dot means differentiation with respect to this argument. The remaining $g_{ik}^{(12)}$ can then be found from the other field equations. One can show by this method—as well as by others—that there is no gravitational dipole wave, the lowest radiative mode being the quadrupole.

The purpose of using two parameters becomes clear at the second approximation. If one uses only one parameter—the mass (or k)—all quadratic products of linear terms, *e.g.*

$$g_{ik}^{(12)} \times g_{ik}^{(13)}, \quad g_{ik}^{(10)} \times g_{ik}^{(12)},$$

are lumped together as the nonlinear part of the second approximation. In fact, such products are of very different orders of magnitude, and mathematical character, and repay separate treatment.

The first interesting nonlinear approximation is the (24) one. This comes from products like

$$g_{ik}^{(10)} \times g_{ik}^{(14)}, \quad g_{ik}^{(11)} \times g_{ik}^{(13)}, \quad g_{ik}^{(12)} \times g_{ik}^{(12)},$$

(and their derivatives), and the second product vanishes because there is no dipole wave so $g_{ik}^{(11)} = 0$ (in any case, from the symmetry of our particularly simple system the octupole terms vanish so $g_{ik}^{(13)} = 0$). The actual equations of the (24) approximation are very complicated, and I was not able to solve it exactly (so I am really speaking of an approximation to an approximation!). However, in the $g_{ik}^{(24)}$, after suitable co-ordinate transformation, one can recognize spherically symmetric terms in r^{-1} with coefficients changing with time. These correspond at great distance to a Schwarzschild solution with diminishing mass, and it turns out that *the rate of decrease of mass is exactly equal to the rate of emission of energy by waves* as determined by the pseudo tensor in the linear approximation.

The $(\log r)/r$ terms which occur in Fock's results turn up also in this work, but only as transients. Therefore, if one examines the distant field of the source a long time after the waves have passed, one finds only the Schwarzschild solution corresponding to the decreased mass.

The double-series method can be used for higher approximations and for more complicated systems. The calculations are formidable but are greatly reduced if one chooses Bondi's form of the metric, described in the next section.

So far, no way has been found of applying the double-series method to free gravitational motion of particles, which is a much more difficult problem than that of forced motion. This is because one has to make sure that the sources are point masses only, and that there are no stress singularities connecting them. It is no longer sufficient to lump all the sources together as an unspecified singularity at the origin, and to examine the far field.

The most determined attack on this problem so far is that of Havas (1957) (see also Havas and Goldberg (1962) and Havas (1964)). Havas starts by taking an energy tensor corresponding to N point masses:

$$(4.18) \quad T^{ik} = \sum_{s=1}^N \int_{-\infty}^{\infty} \beta_{(s)}^{ik}(\tau_{(s)}) \delta^4(x^a - z_{(s)}^a) d\tau_{(s)},$$

where x^a is a world-point, $z_{(s)}^a$ the co-ordinates of the s -th particle, δ^4 the four-dimensional delta function, $\tau_{(s)}$ a parameter along the s -th world line and $\beta_{(s)}^{ik}$ an array of ten functions of $\tau_{(s)}$ to be determined. He uses the identities

$$(4.19) \quad T^{ik}{}_{;k} = 0$$

to evaluate $\beta_{(s)}^{ik}$ and finds

$$(4.20) \quad (-g)^{\frac{1}{2}} \beta_{(s)}^{ik} = M_{(s)} v_{(s)}^i v_{(s)}^k,$$

where $M_{(s)}$ is a quantity connected with the relative mass of the s -th particle and $v_{(s)}^i$ satisfies the geodesic equation

$$(4.21) \quad \frac{dv_{(s)}^i}{ds} + \Gamma_{ab}^i v_{(s)}^a v_{(s)}^b = 0$$

for metric g_{ik} and is to be interpreted as the four-velocity. However, this result is partly formal because g_{ik} is not known in advance.

To determine g_{ik} Havas uses the fast approximation method with the gravitational constant k as a parameter. In the application of the method to forced motion we made no use of (4.19) because we were not then interested in the motion of the source (or if we were we could not find it because all we know about the source is that it is a delta function at the origin). In Havas' work (4.19) is used extensively in the

form of the geodesic eq. (4.21). In the zeroth approximation this gives

$$(4.22) \quad \frac{d\mathcal{V}_{(s)}^i}{ds} = 0$$

since the Γ_{jk}^i vanish in this order, so the particles move in straight lines with constant speed.

This result, first obtained by Lubański (1937), is important for the linear approximation to the field equations,

$$(4.23) \quad G_{ik} + 8\pi T_{ik}^{(0)} = 0.$$

In harmonic co-ordinates this reduces to (4.10) and we now see that the $T_{ik}^{(0)}$ has to be constructed, via (4.20) and (4.18), out of particles moving with constant speed. One would not expect these to radiate (on the electrodynamic analogy), and a calculation with the pseudo tensor confirms this. The linear approximation tells us nothing about energy loss for free gravitational motion.

In proceeding to higher approximations the extreme difficulties of the problem become apparent. One of these is the following. The straight-line motion means that the metric of the first approximation becomes singular along these lines. These singularities will enter the metric of the second approximation through $N_{ik}^{(2)}$ of (4.12), and will enter the higher order equations of motion through Γ_{ab}^i in (4.21). Yet the singularities have no physical meaning, being due merely to the nature of the approximation method. For details of how $T_{ik}^{(0)}$ is modified to overcome the difficulty the reader must be referred to the paper of Havas and Goldberg (1962).

Results about radiative motion are obtained by solving the inhomogeneous wave eq. (4.10) (with suitably modified $T_{ik}^{(0)}$). The solution chosen is the retarded one of Lienard-Wiechert. This is then substituted into (4.21) to obtain the first order equations of motion.

The resulting equations give $\frac{2}{3}$ of the correct value for the planetary perihelion motion; they give also an energy *gain* when applied to the double star problem. As Havas remarks, the equations of motion should be pressed beyond the first order before the results of the method can be properly appraised.

A method, apparently somewhat similar to Havas', for the study of electromagnetic and gravitational radiation damping, has been outlined by C. de Witt (1960, 1963), partly in collaboration with J. L. Ging. A full account is awaited.

5. - Bondi's method.

The method described in this section is based on papers by Bondi and collaborators (Bondi, 1960; Bondi, van der Burg and Metzner, 1962). It is designed to give the distant wave-field of an isolated oscillating system without restriction to freely gravitating bodies, the source being as in the double series method, an unspecified delta function at the origin. The approximation method is to expand the g_{ik} in ascending powers of r^{-1} . No doubt other workers had previously tried this method, and they were probably frustrated by the $(\log r)/r$ terms already mentioned. Bondi avoids them by a judicious choice of co-ordinate system which I now describe.

Suppose that a source of light is placed at a point 0 and surrounded by a small sphere, on which polar angles θ and φ are specified, and also a time co-ordinate u . Bondi then defines the u, θ, φ co-ordinates of an arbitrary event E to be those at which the outward light ray OE cuts the sphere. Along an outward ray u, θ, φ are constant, and a radial variable r is a measure of distance along it. In flat space-time u is equal to $t - r$ and the metric is

$$(5.1) \quad ds^2 = du^2 + 2 du dr - r^2(d\theta^2 + \sin^2 \theta d\varphi^2) .$$

Bondi generalizes (5.1) to the following axially symmetric non flat metric:

$$(5.2) \quad ds^2 = g_{44} du^2 + 2 \exp [2\beta] du dr + 2g_{42} du d\theta - \\ - r^2(\exp [2\gamma] d\theta^2 + \sin^2 \theta \exp [-2\gamma] d\varphi^2) ,$$

where g_{44}, β, g_{42} and γ are functions of u, r and θ . This metric is especially well-suited to the outgoing radiation problem, partly because the retarded time is used as one of the co-ordinates.

Let us number the co-ordinates

$$x_1 = r, \quad x_2 = \theta, \quad x_3 = \varphi, \quad x_4 = u;$$

then of the seven non trivial field equations (for empty space) the following four are called the main equations:

$$(5.3) \quad R_{11} = R_{12} = R_{22} = R_{33} = 0 .$$

The equation $R_{14} = 0$ is satisfied identically if (5.3) are satisfied; the

remaining two,

$$(5.4) \quad R_{24} = R_{44} = 0$$

are called the supplementary conditions.

The structure of the main equations is remarkable. They show that, given γ (see (5.2)) for some value of u , the future development of the system follows, except for the possible intervention of five functions of u and θ which turn up during the integration of the main equations.

Of these five functions one is found to vanish if an outgoing radiation condition is imposed, and one more can be removed by a co-ordinate transformation. Bondi then commences the expansion of the g_{ik} and the field equations in powers of r^{-1} . He finds that two of the remaining functions of integration are determined in terms of a third by imposition of the supplementary conditions. This single remaining function, $c^{(1)}(u, \theta)$, is arbitrary.

To sum up, Bondi shows that, given γ on a light-cone opening out into the future, the system can do something unexpected only by virtue of $c^{(1)}$. He calls $\partial c / \partial u$ the *news function*, and for him the transmission of news by the gravitational field through empty space constitutes gravitational radiation. The function $c^{(1)}$ enters directly into the expansion of γ :

$$\gamma = r^{-1} c^{(1)} + r^{-2} c^{(2)} + \dots$$

We notice that to be given γ on a *future* light-cone is a rather unusual form of initial condition since what we actually have is information about the world only on our *past* light-cone. However, one might argue that, as far as unpredictability is concerned, we could send out our instructions by light-signals (into the future, of course) and these could trigger the components of the system in an assigned way. If one accepts this, one can say that one has the desired information about the system on the future light-cone. It seems implicit in Bondi's view that a freely gravitating system does not radiate since its future is presumably completely determined given sufficient information on the hypersurface $u = \text{const}$.

Consider now a system which oscillates for a finite time between infinite periods of rest. Bondi is able to connect the news function with the change in mass m by the formula

$$(5.5) \quad \frac{dm}{du} = -\frac{1}{2} \int_0^\pi \left(\frac{\partial c^{(1)}}{\partial u} \right)^2 \sin \theta \, d\theta.$$

Thus the mass of the source diminishes when news is transmitted. The result is quantitatively in agreement with that of the double series method, and with that of the linear approximation (plus pseudo tensor).

The result (5.5) is a very important one and the method of getting it is most elegant. There are, though, a number of questions that need answers. First, how does one choose the function $c^{(1)}$ which generates the expansion? One way that suggests itself is the following. Suppose we start from the linearized metric for, say, a quadrupole oscillator, obtained by solving (2.3) and (2.4) and transform it into the form of Bondi's metric (5.2). Then if we pick out the coefficient of r^{-1} in γ we might take this for $c^{(1)}$. Now, if we insert it into Bondi's equations we find that the higher coefficients in the r^{-1} expansion contain polynomials in the time u . Taking g_{44} , for example, we find that at time u , after the oscillations have finished

$$(5.6) \quad g_{44} \sim 1 - \frac{2m}{r} + \frac{au + b}{r^2} + \frac{eu^2 + fu + g}{r^3} + \dots,$$

where a, b, e, f and g are functions of θ . It is not hard to see from Bondi's equations that we may expect to get a similar result if we try any other random choice for $c^{(1)}$.

The appearance of unbounded coefficients of r^{-n} is disquieting. Two questions arise: (i) are the series convergent? (ii) are there approximate solutions of Bondi's type in which the unbounded coefficients do not occur? I shall discuss the second question first.

It may be that for appropriate choice of $c^{(1)}$ the unbounded coefficients do not occur. Indeed, Bondi argues that a « correct » $c^{(1)}$ can be found in principle by solving an infinite set of equations. He is not, of course, able to show that after this procedure we should be left with a non trivial wave-like solution.

What is clear is that if the unbounded coefficients are to be avoided, no finite process is at present known whereby Bondi's approximation method can be put into operation. Of course, approximation procedures are usually infinite processes, but in most cases one can solve each *step* by a finite process, and then proceed to the next, so that with enough trouble one could get up to the n -th stage, for any n . But in Bondi's method this is not possible: an infinite process is needed to solve the first step, namely the r^{-1} terms.

I shall now make a tentative reply to question (i) above. It is known (Bonnor, 1959a) that in certain co-ordinate systems gravitational waves

exhibit tails. Consider a very simple tail for a scalar wave, $\varphi = t^{-1}$, $t \gg 0$ (*). One can, if one wishes, expand this as a series of powers of r^{-1} . Write $u = t - r$ and assume $0 < u < r$; then

$$(5.7) \quad t^{-1} = r^{-1} \left(1 + \frac{u}{r}\right)^{-1} = \frac{1}{r} - \frac{u}{r^2} + \frac{u^2}{r^3} - \dots$$

This has an obvious formal similarity to (5.6) and suggests that series like (5.6) may converge to a wave-tail. Notice, however, that the first term of the series is r^{-1} and might be mistaken for a mass-potential term. This shows that expansions of this type can be misleading.

Another possible explanation for the polynomial terms is the following. The prediction from the linear approximation that an isolated source can transmit momentum by gravitational waves has already been mentioned in Sect. 2. If this happens the source will recoil, and, when emission stops, there will be motion with constant velocity relative to the original rest frame. This motion will cause a secular increase in the dipole and higher moments, just because the distance from the origin is increasing. In this case the series in (5.6) will diverge as $u \rightarrow \infty$, and for an excellent physical reason.

Whilst a good deal of mystery still surrounds the unbounded terms in (5.6), the arguments in the previous paragraphs suggest that they probably have a physical cause. Those who wish to use Bondi's approximation method will have to learn to live with these terms. It is clear, though, that the physical interpretation of these approximate solutions is going to be very difficult.

Let us now reconsider the status of Bondi's eq. (5.5), concerning loss of mass. As Bondi remarks, the result is quite clear cut for a system that starts and finishes in a static state (without tails). However, we are not sure that there are any radiating systems of this type: in Bondi's co-ordinates it may be that all such systems have tails. In that case, his definition of mass would need careful examination; for systems with tails the result cannot yet be regarded as established.

I wish to make it clear that these critical remarks apply to the expansion in powers of r^{-1} , and not to Bondi's metric. The latter is splendidly suited to the study of outgoing radiation from isolated sources, and its discovery (and generalization by Sachs (1962) to the case with no spatial symmetry) is an event of first importance in the theory of gravitational radiation. In a paper in preparation I hope to show that great progress

(*) Of course this does not satisfy the homogeneous wave equation, but this is not relevant. Tails of this sort occur in the work by Bönnor (1959a).

can be made if one uses Bondi's metric in conjunction with the double series method of approximation.

In conclusion, I shall mention briefly the solution of the scalar wave equation which underlies Bondi's work. This equation in the flat space-time of metric (5.1) is

$$(5.8) \quad \varphi_{11} - 2\varphi_{14} + 2r^{-1}(\varphi_1 - \varphi_4) + r^{-2}(\varphi_{22} + \varphi_2 \cot \theta) = 0,$$

axial symmetry being assumed and the co-ordinates being numbered as described previously. This equation has solutions proportional to $P_n(\cos \theta)$:

$$(5.9) \quad \varphi^{(n)} = \psi^{(n)}(r, u) P_n,$$

where

$$(5.10) \quad \psi_{11}^{(n)} - 2\psi_{14}^{(n)} + 2r^{-1}(\psi_1^{(n)} - \psi_4^{(n)}) - r^{-2}n(n+1)\psi^{(n)} = 0$$

and these represent the 2^n -pole waves. Solutions of (5.10) exist which are finite polynomials in r^{-1} ;

$$(5.11) \quad \psi^{(n)} = \sum_k^{(k)} L^{(n)}(u) r^{-k-1}.$$

The coefficients satisfy the recurrence relation

$$(5.12) \quad 2(k+1) \frac{d}{du} L^{(n)} = (n-k)(n+k+1) L^{(n)}.$$

The representation of multipole waves by $\varphi^{(n)}$ is a particularly simple one because the angular part separates. For example, for the quadrupole we have

$$(5.13) \quad \varphi^{(2)} = P_2 \left(2r^{-1} \frac{d^2 Q}{du^2} + 2r^{-2} \frac{dQ}{du} + r^{-3} Q \right);$$

this is to be compared with (4.17).

6. - Conclusion.

It now seems fairly well established that, according to relativity, gravitational waves have a real existence and carry energy. This is the result of work using the fast approximation, and also Bondi's method,

though concerning the latter there are still some reservations, at least in my mind.

Except that it loses energy, almost nothing is known of the role of the source. Historically it is slightly odd that the most difficult problem—that of whether freely gravitating particles radiate—was attempted first. It still represents one of the major challenges of relativity.

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INTERVENTI E DISCUSSIONI

— H. BONDI:

In describing his very interesting and valuable work, Professor Bonnor has made some remarks which indicate a misunderstanding of some of my work, which I will try to clear up in my talk in the afternoon.

— E. NEWMAN:

I am inclined to believe the last of Bonnor's explanations for the occurrence of the polynomials in u within the Bondi metric, namely that they are the consequences of «recoil». By recoil I mean not only a change in the center of mass or of the dipole moment but also a change in the higher moments. An analogous thing occurs in the solution of Maxwell's equations when they are expressed in terms of the characteristic initial data. For example, if there is a quadrupole oscillator at rest initially which begins to oscillate and then returns to rest, the data which governs this is the third derivative of the quadrupole moment, $\ddot{Q}$, (this is the Maxwell theory equivalent of Bondi's news function) which is initially zero, because different from zero, then returns to zero. In general after $\ddot{Q}$ has returned to zero $\ddot{Q} = c$, $\dot{Q} = cu + d$ and $Q = \frac{1}{2}cu^2 + du + e$, etc. It has the same type of polynomial behaviour as found in Bondi's work. I am convinced that this is the origin of these terms.

— G. E. TAUBER:

Recently Synge *et al.* ⁽¹⁾ have developed a systematic method in obtaining Einstein's field equations to any desired degree of approximation. At first sight this method seems to be unsuitable to describe gravitational waves, since the gravitational potentials have been assumed to be stationary. However, it is quite straight-forward to generalize Synge's method to time-dependent cases.

Furthermore, all the solutions to Einstein's field equations—whether they be exact or for a linearized theory—have assumed that at large distances space may be assumed to be flat. Such boundary conditions are only consistent with zero mass; and clearly, no gravitational radiation has any meaning in an universe entirely empty of any matter. A consistent formulation of gravitational radiation in an universe containing some matter, however little or remote, must satisfy the condition of asymptotic approach of all solutions to a consistent wave-free time-dependent world, preferably the approximately homogeneous and expanding universe of experience, as given, for example, by Friedmann ⁽²⁾.

We have therefore assumed that in the zeroth approximation the g_{ij} are given by the Robertson-Walker ⁽³⁾ form of the expanding universe, which have

⁽¹⁾ A. DAS, P. S. FLORIDES and J. L. SYNGE: *Proc. Roy. Soc., A* **263**, 451 (1961).

⁽²⁾ A. FRIEDMANN: *Zeits. f. Phys.*, **10**, 377 (1922).

⁽³⁾ Cf.: A. SCHILD and L. INFELD: *Phys. Rev.*, **68**, 250 (1945).

the nice property that they are conformal to the flat Minkowski line-element. If one then expands the g 's according to:

$$g_{ij} = e^{\Gamma}(\eta_{ij} + \underset{1}{\gamma_{ij}} + \underset{2}{\gamma_{ij}} + \dots),$$

where e^{Γ} corresponds to the underlying metric of the Friedmann universe by the method of Synge one obtains (using the generalized form of the de Donder conditions) wave equations for the various orders. These, of course, are not the of the d'Alembertian form (1.1) but correspond to waves in a Friedmann universe (⁴).

We have found explicit solutions in the first order and are now engaged in obtaining higher order solutions.

One might also mention that the divergencies present in some solutions, such as these one of Rosen and Shanin (⁵) disappear if the boundary conditions of a Friedmann universe is used (⁶).

— P. HAVAS:

Within the framework of the fast motion approximation, the zeroth order equations of motion (which are a consequence of the linearized field equations) are eqs. (4.22), *i.e.* zero acceleration. The first order equations of motion contain radiation reaction terms if retarded interactions are assumed. These equations were applied to the two-body problem by S. F. Smith and myself. We took the point of view that solution should be sought with the best available techniques, treating the equations as if they were exact, postponing a consideration of the limitations due to the approximate nature of the equations until such a solution was obtained. It was found that the radiation reaction terms produced an *outward* spiraling of the two particles, which can be interpreted as an energy gain by all reasonable definitions of the energy of the system, and independently of any particular definition of an energy-momentum pseudo-tensor of the field. As I noted yesterday after Mme. Tonnelat's talk, a similar result is obtained if a special relativistic linear theory of gravitation is used. Nevertheless, we do not believe that this result is conclusive. It may well be that the equations of motion obtained in higher orders in the fast approximation method will lead to an energy loss, or possibly to no radiation effects at all; these questions are being investigated, but the mathematical difficulties are considerable.

On the other hand, we do believe that none of the papers claiming to show that a two-particle system must experience an energy loss has established this point. For work based on the EIH method this has been discussed by Dr. Bonnor. Most other work in essence follows the procedure of Einstein's paper of 1918, based on the linearized theory, calculating an energy flow to infinity and relating it to the variation of the quadrupole moment of the source,

(⁴) N. ROSEN and G. E. TAUBER: *Bull. Res. Council of Israel*, 9 F, 111 (1960).

(⁵) N. ROSEN and H. SHANIN: *Rev. Mod. Phys.*, 29, 429 (1957).

(⁶) G. E. TAUBER and N. ROSEN: (to be published).

by manipulations with the pseudotensor, without realizing that the field equations employed imply non accelerated motion.

— H. BONDI:

The reality of energy transfer across empty space ($T_{\mu\nu} = 0$) by gravitation can be demonstrated even in Newtonian theory. Consider two bodies, A and B , being each axially symmetric and symmetric about their common equatorial plane. Each has machinery inside driven by an electric storage battery that can change its shape from prolate through spherical to oblate and back. Whenever A is oblate, B is prolate and viceversa such that the attraction between the two bodies is always exactly as though they both were spheres. Accordingly they describe Kepler orbits about each other, orbits which we shall suppose to be ellipses of large eccentricity. Each body produces tideraising forces on the other, which tend to make it oblate and these forces are highly distance dependent. A becomes oblate whenever B is near, thus gaining a good deal of energy, becoming prolate again when B is far apart so that little energy is needed for the change of shape. B , going through the opposite cycle, has the opposite experiences. Thus in each cycle the energy content of A 's battery is increased and that of B diminished by the same amount. Therefore energy has been transferred from B to A across empty space.

— W. B. BONNOR:

I wish to reply only to Professor Newman's remarks. I am quite sure that one of the causes of the polynomial terms is that stated by him. I am equally sure that there are other causes—for example, the existence of wave tails in Bondi's co-ordinates.

[Alla discussione partecipò anche G. McVITTIE.]

Gravitational Waves.

H. BONDI

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1. - Introduction.

If we consider an isolated source, then the changes of its gravitational field according to Newtonian theory are limited by the law of conservation of mass (no change in the monopole field) and of conservation of momentum (linear time-dependence of dipole field, thanks to the non-existence of negative masses). The higher multipoles depend only on the shape of the source and may, therefore, be arbitrary functions of the time. Accordingly these changes of shape enable signals to be transmitted from the source to a distant observer. In Newtonian theory these signals naturally travel with infinite speed. In any relativistic theory of gravitation this is unacceptable, and we cannot envisage the speed at which gravitational information travels as exceeding the speed of light. Moreover, viewed in this way we find also of relevance the basic assumption of modern physics that it is impossible to transmit information without transmitting energy. Unhappily no rule exists that states how much energy corresponds to any given amount of information.

If we accept the equality not just of energy and mass in general, but of energy and active gravitational mass (that is, gravitation-producing mass), then the transportation of energy from the source must involve a diminution of the monopole field. This shows immediately in a perhaps slightly different way from the usual that any sensible relativistic theory of gravitation must be nonlinear, for though the monopole field cannot, as it were, of its own accord vary, fluctuations in the higher multipole fields must imply a diminution in the monopole field. Again, with the natural assumption that only outgoing waves occur, the time derivative of the monopole term must be nonpositive, which again can only arise from a quadratic appearance of the changes of the higher fields. Thus, the nonlinearity of general relativity, which has prejudiced some against it, is not an accidental feature of this particular relativistic formulation of gravitational theory, but a necessary feature.

2. - Mathematical formulation.

By Birkhoff's theorem there can be no time variation of a spherically symmetrical empty space field in general relativity. Accordingly the highest degree of symmetry we can allow for is axial symmetry, and hence at least three co-ordinates must enter the metric. There seems to be no hope of finding an exact solution of the field equations for such a complex situation. Since an approach via the linear approximation will hide the essential nonlinear features, an alternative approach has been chosen, namely an asymptotic one. We look at the field of an isolated system from a great distance. This has the advantage of rendering many features clear and, in particular, of confining the considerations to the empty space area. It is also known from other wave theories (Maxwell) that the radiation field, as opposed to the near field, only becomes clear at large distances. On the other hand, such an approach makes it *a priori* impossible to see in any detail how the matter of the source interacts with the field to produce excitation. The first step in the formulation must be the introduction of a co-ordinate system suitable for this asymptotic formulation. As we have specified outgoing waves only, we choose our co-ordinates so that three of them remain constant on outward travelling light rays. Two of these co-ordinates will be angular and one will be time-like. The remaining co-ordinate is radial in nature and is chosen so that the weakening of the wave with distance from the source is adequately represented. The choice is therefore made so that the element of area enclosed by a bundle of rays is proportional to the square of this radial co-ordinate. On this basis the metric may be put in the form

$$ds^2 = \left(\frac{v}{r} \exp[2\beta] - r^2 U^2 \exp[2\gamma] \right) du^2 + 2 \exp[2\beta] du dr + \\ + 2r^2 U \exp[2\gamma] du d\theta - r^2 (\exp[2\gamma] d\theta^2 + \exp[-2\gamma] \sin^2 \theta d\varphi^2) .$$

One of the difficulties of the problem shows itself here. We specify the co-ordinates asymptotically at large distance from the source, which of course in general will itself have finite extension. Therefore it will be impossible to specify a unique bundle of rays that at infinity represents these waves. Accordingly there is a good deal of freedom in the system of co-ordinates described by a peculiar law of transformations. This law may be subdivided into two kinds of transformations, one which corresponds to motion of the co-ordinate system relative to the source (the Lorentz type transformation) and one which corresponds to the angular disposition of the light rays. Note that, in general, if the light rays are

pursued inwards too far they will intersect, but this is no problem in an asymptotic formulation.

From this metric field equations may be derived which, through the operation of the identities, take a peculiarly simple form. Four of the equations, $R_{33} = R_{11} = R_{12} = R_{22} = 0$, are independent of each other and are called the main equations. Four drop out entirely (three through axial symmetry, one as a consequence of the identities), and the remaining two take the form that only the r^{-2} term is not already made zero by the main equations. These two conditions, which therefore depend only on two co-ordinates, we call the supplementary equations.

We are now ready to apply the condition that only outward travelling waves are present. This takes the form that all the terms in the metric can be expanded in negative powers of the radius, the co-efficients being functions of the time-like and angle variables only. To put it more precisely, all that is required is that each co-efficient should be expressible as the sum of the first three or four negative powers of the radius, together with a remainder term vanishing more rapidly as the radius increases than the last term included. Three of the main equations do not involve time differentiation, but allow each of the co-efficients to be deduced on a light cone from the knowledge of one of the co-efficients of the metric.

The fourth equation then determines the time variation of this basic co-efficient. There are five functions of integration involved in this procedure, two of which, however, may be reduced to zero by co-ordinate conditions. The supplementary conditions then establish two links between the three surviving functions. One of these is an arbitrary function of the time and angle co-ordinates and conveys the information about the behaviour of the source. This is accordingly called the news function. The second co-efficient is called the mass aspect and its angular mean value is called the mass. In any static situation (in which alone there has previously been a clear definition of mass) the mass so specified agrees with the earlier definitions. The last of the three functions of integration represents the dipole moment. The relations have the consequence that any emission of news leads to a diminution of mass. This is the central result of the work. In suitable circumstances the emission of news can also affect the dipole moment so that, as it were, a gravitational wave rocket action is conceivable.

It is interesting to note that a news-free system need not be static. It does not radiate information and therefore does not lose mass, but may well have time variability. However, it is not clear what motions of the source correspond to such news free time-varying situations. The only case clearly identified is a Schwarzschild mass in uniform motion relative to the system of co-ordinates. This inability to identify the motions of the source is perhaps the greatest limitation of an asymptotic theory.

It is not correct to say that there is a delta function at the origin in my method. The asymptotic treatment cannot be carried far upwards because the intersection of light rays makes the co-ordinate system break down. No doubt many different distributions of matter would give the same asymptotic field, and a few of them may be chosen to be singular.

Next, I simply do not know whether a freely moving system of bodies radiates or not; my work does not imply any particular answer to this most interesting question.

Finally, we can certainly switch off the radiation and so the rate of mass loss by making the news function vanish, so the difficulty suggested by Bonnor does not occur. What does happen is that in general the system in its final nonradiating state is not static.

INTERVENTI E DISCUSSIONI

— E. NEWMAN:

I wish to present and discuss some proposed definitions (by A. Janis and myself to appear in the *Journ. of Math. Phys.*) of multipoles in general relativity. I must apologize for the fact that all equations are correct modulo signs factors of 2 and $\sqrt{2}$ as I do not have my notes with me.

Probably the simplest way to begin is to start with Maxwell theory and then state without proof the analogous results in gravitational theory.

In flat space introduce the family of null cones around the origin labelling each cone by the parameter u (the retarded time) and introduce r , ϑ and φ as usual. This is the co-ordinate system used by the two previous speakers. Also introduce four independent vector fields in the following fashion; the null vector field l^μ tangent to the null cones, the null field n^μ pointing in towards the origin (tangent to the past cone) and two complex vectors m^μ and its complex conjugate $\bar{m}^\mu$ obtained from two unit orthogonal space-like vectors a^μ and b^μ by

$$m^\mu = \frac{1}{\sqrt{2}} (a^\mu + ib^\mu).$$

These four vectors form our local tetrad system. Using these vectors one can define:

$$\varphi_0 \equiv F_{\mu\nu} l^\mu m^\nu, \quad \varphi_1 \equiv \frac{1}{2} F_{\mu\nu} (l^\mu n^\nu + \bar{m}^\mu m^\nu), \quad \varphi_2 \equiv F_{\mu\nu} \bar{m}^\mu n^\nu$$

the three complex φ 's replacing the six $F_{\mu\nu}$'s.

One can now write down Maxwell's equations in terms of these φ 's and then solve them directly. Rather than give the general solution I will write down some well known specific solutions (modulo numerical factors).

— *Monopole or Coulomb:*

$$\varphi_0 = 0, \quad \varphi_1 = \frac{e}{r^2}, \quad \varphi_2 = 0.$$

— *Radiating dipole:*

$$\varphi_0 = \frac{p \sin \theta}{r^3}, \quad \varphi_1 = \frac{\dot{p} \cos \theta}{r^2} + \frac{p \cos \theta}{r^3}, \quad \varphi_2 = \frac{\ddot{p}(u) \sin \theta}{r} + \frac{\dot{p} \sin \theta}{r^2} + \frac{p \sin \theta}{r^3},$$

a real p giving an electric type pole an imaginary p , a magnetic type pole.

— *Radiating quadrupole:*

$$\begin{aligned} \varphi_0 &= \frac{\dot{Q} \sin \theta \cos \theta}{r^3} + \frac{Q \sin \theta \cos \theta}{r^4}, \\ \varphi_1 &= \frac{\ddot{Q}}{r^2} \left(1 - \frac{1}{3} \cos^2 \theta \right) + O\left(\frac{1}{r^3}\right), \\ \varphi_2 &= \frac{\ddot{Q}}{r} + O\left(\frac{1}{r^2}\right), \end{aligned}$$

Q being the quadrupole moment.

It is easy to generalize this to an arbitrary pole or collection of poles. An important thing to note is that the coefficient of the leading term in φ_2 is an arbitrary function of u , being completely analogous to Bondi's news function.

One can now obtain very analogous results in general relativity (asymptotically for large r) by first defining the tetrad components of the Weyl tensor (analogous to the φ 's in Maxwell theory) by

$$\begin{aligned} \Phi_0 &= C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma l^\delta, \quad \Phi_1 = C_{\alpha\beta\gamma\delta} l^\alpha n^\beta m^\gamma l^\delta, \\ \Phi_2 &= \frac{1}{2} C_{\alpha\beta\gamma\delta} (l^\alpha n^\beta n^\gamma l^\delta + l^\alpha n^\beta m^\gamma \bar{m}^\delta), \quad \Phi_3 = C_{\alpha\beta\gamma\delta} l^\alpha n^\beta n^\gamma \bar{m}^\delta, \quad \Phi_4 = C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta \bar{m}^\gamma n^\delta. \end{aligned}$$

In general one can prove (with certain very simple asymptotic assumptions including the assumption that $\Phi_0 = O(1/r^6)$) that

$$\begin{aligned} \Phi_1 &= \frac{\Phi_1^0}{r^4} + O\left(\frac{1}{r^5}\right), \quad \Phi_2 = \frac{\Phi_2^0}{r^3} + O\left(\frac{1}{r^4}\right), \\ \Phi_3 &= \frac{\Phi_3^0}{r^2} + O\left(\frac{1}{r^3}\right), \quad \Phi_4 = \frac{\Phi_4^0}{r} + O\left(\frac{1}{r^2}\right). \end{aligned}$$

As a special case the Schwarzschild solution has the form:

$$\Phi_0 = \Phi_1 = \Phi_3 = \Phi_4 = 0 \quad \text{and} \quad \Phi_2 = \frac{m}{r^3}.$$

In the outgoing radiation case, *i.e.*

$$\Phi_0 = \sum_{n=0}^L \frac{\Phi_0^n}{r^{5+n}} + O\left(\frac{1}{r^{5+L}}\right),$$

one can prove that to avoid angular singularities in the solution we must have the following angular behaviour; (we have assumed axial symmetry but have allowed rotation)

$$\begin{aligned} \Phi_4^0 &= \sum_{n=2}^{\infty} A_n(u) P_n^2(\cos \theta), & \Phi_3^0 &= \sum_{n=1}^{\infty} B_n(u) P_n^1(\cos \theta), & \Phi_2^0 &= \sum_{n=0}^{\infty} C_n(u) P_n^0(\cos \theta), \\ \Phi_1^0 &= \sum_{n=1}^{\infty} D_n(u) P_n^1(\cos \theta), & \Phi_0^m &= \sum E_n^m P_n^2(\cos \theta). \end{aligned}$$

The moments can now be defined as follows:

$$\text{Monopole or mass} \quad m = -\frac{1}{4} \int_0^\pi (\Phi_2^0 + \bar{\Phi}_2^0) P_0^0(\cos \theta) \sin \theta \, d\theta.$$

Dipole moment $\equiv$ mass dipole + i angular momentum =

$$= -\frac{1}{2} \int_0^\pi \Phi_1^0 P_1^1(\cos \theta) \sin \theta \, d\theta.$$

$$\text{Quadrupole moment} = -\frac{1}{2} \int_0^\pi \Phi_0^0 P_2^2(\cos \theta) \sin \theta \, d\theta.$$

Higher moments as well as the linear momentum can be easily defined.

These moments agree with those obtained from a bounded source when the linearized theory is used.

There is a simple relation between the moments defined here and those obtained by Bondi. For example,

$$m_B = m - \frac{1}{4} \int_0^\pi \frac{\partial}{\partial u} (C\bar{C}) \sin \theta \, d\theta,$$

where m_B is Bondi's mass and $\dot{C}(u)$ is Bondi's news function.

Bondi's energy conservation law

$$\frac{\partial m_B}{\partial u} = -\frac{1}{2} \int_0^\pi \dot{C}\dot{\bar{C}} \sin \theta \, d\theta$$

becomes

$$\frac{\partial m}{\partial u} = -\frac{1}{2} \int_0^\pi \dot{C}\dot{\bar{C}} \sin \theta \, d\theta + \frac{1}{4} \int_0^\pi \frac{\partial^2}{\partial u^2} (C\bar{C}) \sin \theta \, d\theta$$

which is another perfectly acceptable conservation law.

We feel that these definitions are better than Bondi's because they are more invariantly defined via the Weyl tensor and hence they can be directly measured (in principle) by the methods of geodesic deviation.

It appears possible that one can give meaning to both masses m and m_B as for example m might be the (bare) mass of the source alone while m_B might be the total mass including radiation field energy. This is just a conjecture.

— H. BONDI:

We have a clear understanding of the term « mass » only in the static and stationary cases, so that any definition of the term in the dynamic situation is acceptable provided only it reduces to the correct expression in static and stationary conditions. As against the evident advantages of Newman's definition, mine may yet be preferable because of its monotonic variation with time.

— A. PAPAPETROU:

Starting from the properties of shock waves one can prove a theorem which gives useful information about radiation.

Consider firstly the Maxwell theory (Minkowski space) and a shock wave of order two:

$$[A_{\lambda\nu\mu}] = \varphi_\lambda P_\nu P_\mu$$

φ_λ discontinuity « vector », P_μ normal to the characteristic hypersurface. Besides the local condition there is a propagation relation which has the form

$$(\varphi_\lambda \varphi^\lambda P^\mu)_{,\mu} = 0.$$

Now for a hypersurface Σ tending to a light cone one has $P^\mu_{,\mu} = 2/r + \dots$ and this relation gives

$$(1) \quad \varphi_\lambda \varphi^\lambda \sim \frac{1}{r^2} \quad \text{for } r \rightarrow \infty \rightarrow \varphi^\lambda \sim \frac{1}{r}.$$

Consider then a field which is stationary below Σ , therefore $F_{\mu\nu} \sim 1/r^2$ below Σ . It follows at once from (1) that we shall necessarily have $\dot{F}_{\mu\nu} \sim 1/r$ above Σ . Conversely, if $\dot{F}_{\mu\nu}$ is stationary below Σ , and there is no term $\sim 1/r$ in $\dot{F}_{\mu\nu}$ also above Σ there will be no shock wave on Σ (or on any hypersurface above Σ). Therefore, excluding pathological functions, one concludes that the field which is stationary below a hypersurface Σ and has no term $\sim 1/r$ also above Σ will necessarily be stationary also above Σ . Exactly the same reasoning can be repeated for the gravitational field and therefore the same result holds: if a gravitational field is stationary below Σ and has a shock wave on Σ , its curvature tensor will necessarily contain a term $\sim 1/r$ above Σ . Conversely, if the field is stationary below a hypersurface Σ and its curvature tensor has no term $\sim 1/r$ also above Σ , there will be no shock wave on Σ or on another hypersurface above Σ . Therefore, again excluding pathological functions, one concludes that the field will be stationary also above Σ .

Since the existence of terms $\sim 1/r$ in $F_{\mu\nu}$ or $R_{\mu\nu\sigma\alpha}$ is a necessary and sufficient condition for the existence of electromagnetic or gravitational radiation, we have the following result: in the Maxwell as well as in the Einstein theory — and certainly also in the Einstein-Maxwell theory — a system which was initially stationary will necessarily emit radiation (electromagnetic or gravitational or electromagnetic *and* gravitational) from the moment it ceases to be stationary. It is to be stressed that in the previous argument no assumption has been made about a gravitational or nongravitational character of the forces which caused the departure from stationarity.

It is therefore, from this point of view, unjustified to expect that purely gravitational (accelerated) motions will not be followed by an emission of gravitational radiation.

— C. MØLLER:

To Newman's very interesting work in which he defines a mass of the system which differs from that introduced by Bondi, I would like to make the following remarks. In order that a quantity can be interpreted as a mass or energy it should together with the linear momentum form a four-momentum P_i which for a nonradiative system behaves like a 4-vector at least under transformations which are asymptotically Lorentzian.

As shown in my paper Bondi's expression for the mass $m(u)$ follows from a consistent expression for the energy-momentum complex which even makes P_i a free vector under arbitrary space-time transformations. Also I am somewhat uneasy about the fact that the mass defined by Newman does not seem to be steadily decreasing, although the solution considered contains only outgoing waves.

— W. B. BONNOR:

Following Bondi's remarks I should like to clarify certain points in my report.

In regard to the source in BVM's metric, Bondi assures me that it is not a delta function at the origin, because the metric becomes singular for certain values of r greater than zero. I am glad to know this, but it does not alter the substance of my remark that the source in BVM's metric is not specified.

Concerning free gravitational motion, my interpretation of Bondi's view was based on two premises. First, that Bondi regards gravitational radiation as synonymous with the propagation of news. Secondly, the conjecture that free gravitational motion is determined given appropriate initial conditions; if this is correct, there is no news in gravitational motion, and so, on Bondi's view, no radiation.

Finally, I want to stress again that the occurrence of unbounded coefficients in BVM's work cannot be brushed aside as unimportant. There are several possible explanations for them and further progress seems to me conditional on finding which are correct.

— G. E. TAUBER:

I should like to clarify the question of boundary conditions raised earlier. I did not imply that there cannot be any radiation in an empty space.

However, as soon as we have a receiver—suitably constructed by Bondi's engineer—the universe is no longer empty. It is also worth noting that the homogeneous expanding solutions include those asymptotic to a practically empty world, for which the metric approaches the Lorentz-Minkowski form of special relativity at a later date in the history of the expansion. Though at first sight, it may seem that wave solutions here may resemble those with empty space boundary conditions, a closer study shows that in an expanding universe there are always regions of space-time corresponding to distant reaches of space, like the distant part, where curvatures are far from negligible. Hence only certain types of time-limited signals escape the influence of that curvature at the boundary. Alternately, a signal unlimited in duration finds its boundary conditions in a strongly curved space. And it is for that reason why we are proposing the expanding universe boundary conditions.

— H. BONDI:

I have much sympathy with this point of view. Indeed I started my own investigation with the aim of disproving the possibility of gravitational radiation into empty space, thinking that the theory would require an absorber of the Wheeler-Feynman-Hogarth type. As you know, I found instead that the theory admits solutions radiating into empty space, but certainly the important and very difficult problem of wave solutions with cosmological boundary conditions deserves attention.

— P. G. BERGMANN:

On the basis of the linearized theory I suggest that the class of news functions without unbounded tails is described by (otherwise arbitrary) functions $Q(u)$ (amplitudes of quadrupole moments etc.) such that

$$[Q(u)]_{u \rightarrow -\infty} = Q^{(-)}, \quad [Q(u)]_{u \rightarrow +\infty} = Q^{(+)}.$$

This class of functions is at least equivalent to the class of functions of continuity C^s on a finite interval $0 \leq u \leq 1$.

— S. DESER:

The Bondi metric seems to have been used as the universal touchstone of all relativistic techniques. Let me therefore mention here another investigation carried out in collaboration with M. J. Levin, which analysed the Bondi solution in terms of the initial value problem. One looks at the expansion of the metric at constant t , large r ($r \sim t$) rather than in terms of given u at large r . In this way, the results of the canonical method in the wave zone (Arnowitt, Deser, Misner) can be applied and tested on this explicit radiating solution. One finds very nice agreement between the physical interpretation given by Bondi and Sachs and the corresponding canonical variables. Thus the news function is just the unconstrained free part of the spatial metric (the « transverse-traceless » part); the mass-loss agrees with Bondi's negative-definite form (but not with Newman's, therefore) and so on.

With respect to the polynomial behaviour, in u , one possibility (which is not entirely firm) is that, from the initial condition point of view, it may arise

as follows: if one assumes that the *functional form* of the news part is $1/t$, then as a result of the 3R constraint equation, the constraint component g^t of the metric satisfies in the wave zone an equation of the form $\nabla^2 g^t \sim 1/r^2$ and consequently has a contribution $\sim (\ln r)/r$.

This may be rewritten as $\ln(u-t)/r$ and if one expands the logarithm at small u for $t \sim r$, one indeed obtains a polynomial expansion as the price of avoiding an explicit logarithmic part such as has been found in other radiative solutions. Of course, this behaviour is purely in the wave zone; beyond the wave everything falls off nicely since the mass is finite.

— A. H. TAUB:

It is difficult for me to see the analogy between the electromagnetic case and the gravitational one since in the former case the sources are defined in terms of the derivations of the field strength and in the latter case they are defined in terms of the components of the Einstein tensor.

The curvature tensor must satisfy the Bianchi identities but these identities are the integrability conditions for the existence of an affine connection and have no particular reference to the Einstein equations.

— J. ELLERS:

I shall try to answer Prof. Taub's question concerning the connection between the multipole moments of the source and the coefficients of the $1/r$ powers in the series development of the conformal curvature tensor just described by Professor Newman. In the linearized theory this relation is obvious: one writes the deviation of the metric from the flat metric as a retarded integral over the source and develop the integral in powers of $1/r$ choosing some origin within the source region. Making use of the conservation law for the energy-momentum one expresses the above-mentioned coefficients by multipole moments and their (retarded) time derivatives respectively. For not too strong fields this procedure can also be used in the full theory if Einstein's field equation with outgoing radiation condition is rewritten, in harmonic coordinates, as an integrodifferential equation for $g^{\alpha\beta}$ [cf. PERES: *Nuovo Cimento*, 27, 1266 (1963)].

Champ de Dirac, champ du neutrino et transformations $\mathcal{C}$, $\mathcal{P}$, $\mathcal{T}$ sur un espace-temps courbe.

A. LICHNEROWICZ

Collège de France - Paris

Je me propose, dans cette conférence, d'étudier la théorie quantique des champs spinoriels de spin $\frac{1}{2}$ sur un espace-temps courbe. Je me limiterai ici aux champs libres pour lesquels une théorie mathématique rigoureuse peut être développée.

I - LES SPINEURS SUR UN ESPACE-TEMPS COURBE

I. - La variété espace-temps.

a) Soit V_4 une variété différentiable de dimension 4, de classe C^∞ . Nous supposons l'espace-temps V muni d'une métrique riemannienne ds^2 de signature $+- - -$. En chaque point x de V_4 l'équation $ds^2 = 0$ définit dans l'espace vectoriel T_x tangent en x le cône élémentaire C_x .

Nous rapportons systématiquement V_4 à des repères orthonormés, éléments d'un *espace fibré principal* $\mathcal{E}(V_4)$ sur V_4 de groupe structural le groupe de Lorentz homogène $L(4)$. Relativement à ces repères $y = \{e_0, e_A\}$ ($A = 1, 2, 3$), la métrique s'écrit localement sur un ouvert de V_4 :

$$(1.1) \quad ds^2 = g_{\alpha\beta} \theta^\alpha \theta^\beta \quad (\alpha, \beta, \dots = 0, 1, 2, 3)$$

où les θ^α sont des formes de Pfaff et où $g_{\alpha\beta} = \eta_{\alpha\beta}$ ($\eta_{\alpha\beta} = 0$ pour $\alpha \neq \beta$, $\eta_{00} = 1$, $\eta_{AA} = -1$).

b) Le groupe $L(4)$ comprend 4 composantes connexes. Si $A = (A_\alpha^i) \in L(4)$, nous pouvons définir son appartenance à l'une des 4 composantes en introduisant les éléments suivants: nous appelons *signature totale* ε_A de A un nombre égal à ± 1 selon le signe de $\det A$. La *signature*

temporelle ϱ_A est égal à ± 1 selon le signe de A_0' ; enfin la *signature spatiale* σ_A est le produit $\varepsilon_A \varrho_A$.

Nous appelons *orientation totale* ε de V_4 un objet géométrique défini sur V_4 , relativement aux $y \in \mathcal{E}(V_4)$ par une composante $\varepsilon_y = \pm 1$ telle que si $y = y'A$

$$\varepsilon_y = \varepsilon_{y'} \varepsilon_A.$$

V_4 admet au plus 2 orientations totales ε et $-\varepsilon$, mais peut ne pas en admettre. Si V_4 est munie de l'orientation ε , considérons un recouvrement de V_4 par des voisinages munis de repères orthonormés. Les

$$\varepsilon \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$$

définissent sur V_4 une 4-forme globale η , l'élément de volume associé à la métrique et à l'orientation.

Une *orientation temporelle* ϱ est définie sur V_4 , relativement aux $y \in \mathcal{E}(V_4)$ par une composante $\varrho_y = \pm 1$ telle que si $y = y'A$

$$\varrho_y = \varrho_{y'} \varrho_A$$

Nous supposons explicitement que V_4 admet une orientation totale ε et une orientation temporelle ϱ que nous choisissons.

Un vecteur e_0 (avec $e_0^2 = 1$) est *orienté vers le futur* (resp. *passé*) si la composante de ϱ relativement aux repères orthonormés (e_0, e_A) est 1 (resp. -1). L'orientation temporelle permet ainsi de distinguer de façon cohérente, sur toute la variété les demi-cônes de C_x *demi-cône futur* C_x^+ et *demi-cône passé* C_x^- .

La donnée de ε et ϱ définit sur V_4 une *orientation spatiale* $\sigma = \varepsilon \varrho$. Ces différentes orientations jouent un rôle fondamental pour la définition sur un espace-temps courbe des transformations $\mathcal{P}$ et $\mathcal{T}$.

2. - Variétés globalement hyperboliques.

a) Sur V_4 munie de ϱ , un *chemin temporel* est un chemin dont la tangente en tout point x est dans ou sur C_x . Si K est un ensemble de V_4 , le futur $\mathcal{E}^+(K)$ est l'ensemble des chemins temporels issus des points x de K dans le futur de x . Définition analogue pour le *passé* $\mathcal{E}^-(K)$.

Ces définitions s'appliquent en particulier à $K = \{x'\}$ ($x' \in V_4$). L'émission $\mathcal{E}(x')$ est la réunion $\mathcal{E}^+(x') \cup \mathcal{E}^-(x')$. La frontière $\Gamma_{x'}$ de cette émission est le *conoïde caractéristique* de sommet x' . Ce conoïde est lieu de géodésiques isotropes issues de x' . Au voisinage de son sommet $\Gamma_{x'}$ est difféo-

morphe à un voisinage de sommet d'un cône. Il n'en est plus de même loin de x' ; en particulier les géodesiques isotropes issues de x' peuvent se recouper.

b) La théorie des systèmes hyperboliques à coefficients variables conduit à introduire l'hypothèse suivante qui assure l'existence de solutions élémentaires, même en présence de singularités du conoïde.

L'espace-temps V_4 est supposé globalement hyperbolique, autrement dit l'ensemble des chemins temporels joignant deux points est toujours soit vide, soit compact. Sur une telle variété, un ensemble K est dit *compact vers le passé* si $K \cap \mathcal{E}^-(x)$ est compact ou vide pour tout x . Si K est compact vers le passé et K' compact, $\mathcal{E}^+(K) \cup \mathcal{E}^-(K')$ est compact. Tout point d'une variété localement hyperbolique admet un voisinage Ω homéomorphe à un boule ouverte et globalement hyperbolique de telle sorte que les résultats précédents sont valables sur Ω .

3. - Le groupe spinoriel.

Nous désignons par $\text{spin}(4)$ le groupe spinoriel revêtement d'ordre 2 du groupe de Lorentz $L(4)$. Soit $\{\gamma_a\} = \{\gamma_{ab}^a\}$ un système de matrices (4×4) de Dirac que nous prendrons réelles (voir Appendice) (*)

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = -2\eta_{\alpha\beta} e \quad (\gamma^\alpha = \eta^{\alpha\beta} \gamma_\beta).$$

Soit p la projection canonique de $\text{spin}(4)$ sur $L(4)$. Le groupe $\text{spin}(4)$ peut être défini au moyen de *matrices réelles* (4×4) , A , avec $\det A = 1$; si $A \in \text{spin}(4)$, il lui correspond par p la matrice $A = pA \in L(4)$ telle que

$$(3.1) \quad A\gamma_\alpha A^{-1} = A_\alpha^{\lambda'} \gamma_{\lambda'}.$$

Entre les algèbres de Lie $\mathcal{A}(\text{spin}(4))$ et $\mathcal{A}(L(4))$, on a les isomorphismes définis par:

$$(3.2) \quad p': \quad \lambda\gamma^\alpha - \gamma^\alpha\lambda = -\mu^{\alpha\beta}\gamma_\beta, \quad p'^{-1}: \quad \lambda = -\frac{1}{4}\mu^{\alpha\beta}\gamma_\alpha\gamma_\beta$$

où $\lambda \in \mathcal{A}(\text{spin}(4))$ et où $\mu^{\alpha\beta}$ est le tenseur antisymétrique définissant $\mu \in \mathcal{A}(L(4))$. Nous désignons par $\sim$ le passage d'une matrice à son adjointe. On démontre qu'il existe une matrice imaginaire pure β , vérifiant $\tilde{\beta} = \beta$ et $\beta^2 = e$, telle que:

$$(3.3) \quad \beta\gamma_\alpha\beta^{-1} = -\tilde{\gamma}_\alpha.$$

(*) Les indices grecs sont dits indices tensoriels, les indices latins $a, b = 1, 2, 3, 4$ indices spinoriels.

On en déduit que quelle que soit $A \in \text{spin}(4)$:

$$(3.4) \quad A^{-1} = \varrho_A \beta^{-1} \tilde{A} \beta$$

où ϱ_A est la signature temporelle de la matrice pA .

4. - L'espace fibré des repères spinoriels.

a) Nous faisons l'hypothèse suivante: du fibré principal $\mathcal{E}(V_4)$ des repères orthonormés, on peut déduire *par extension* un fibré principal $\mathcal{S}(V_4)$ de groupe structural $\text{spin}(4)$. Lorsqu'il en est ainsi, nous dirons que V_4 admet une *structure spinorielle*; un point z de $\mathcal{S}(V_4)$ est un *repère spinoriel* et $\mathcal{S}(V_4)$ l'*espace fibré des repères spinoriels*. On désigne par $\mathcal{S}_x$ l'espace des 1-spineurs $\psi = (\psi^a)$ contrevariants en x , par $\mathcal{S}'_x$ celui des 1-spineurs $\varphi = (\varphi_a)$ covariants, avec

$$(4.1) \quad \psi(\zeta A^{-1}) = A \psi(\zeta), \quad \varphi(\zeta A^{-1}) = \varphi(\zeta) A^{-1} \quad (A \in \text{spin}(4))$$

Par produit tensoriel de $\mathcal{S}_x$, $\mathcal{S}'_x$ et du complexifié $T_x^{\mathbb{C}}$ de l'espace tangent en x , on obtient la notion de tenseur-spineur. En particulier les γ_{ab}^a sont les composantes, par rapport à un repère spinoriel arbitraire, d'un vecteur-2-spineur γ , le *vecteur-spineur fondamental* de V_4 .

b) Nous notons $\mathcal{C}$ la *conjugaison de charge*, application antilinéaire de $\mathcal{S}_x$ sur lui-même, définie ici par (*)

$$\mathcal{C}: \psi \rightarrow \psi^c = \psi^*$$

$\mathcal{A}$ est l'*adjonction de Dirac*, application antilinéaire de $\mathcal{S}_x$ sur $\mathcal{S}'_x$ définie par:

$$\mathcal{A}: \psi \rightarrow \bar{\psi} = \varrho \tilde{\psi} \beta$$

$\mathcal{A}$ et $\mathcal{C}$ s'étendent naturellement à des tenseurs-spineurs arbitraires et anticommuteurs sur les 1-spineurs.

c) En théorie quantique des champs, on substitue aux composantes complexes d'un spineur ψ des composantes opérateurs linéaires dans un espace de Hilbert $\mathcal{H}$. On note par le symbole $\#$ le conjugué hermitien d'un opérateur de $\mathcal{H}$ et par le symbole $\times$ le produit de la conjugaison hermitienne $\#$ par la transposition en tant que matrice d'opérateurs.

(*) On note $*$ la conjuguée complexe d'une matrice.

Dans ce cas :

$$\psi^c = \psi^\sharp, \quad \bar{\psi} = \varrho \psi^\times \beta.$$

5. - Le 2-spineur ξ .

Il existe un isomorphisme naturel S de l'espace des formes sur l'espace des $(1, 1)$ spineurs. De la forme élément de volume η déduisons le $(1, 1)$ -spineur imaginaire pour :

$$(5.1) \quad \xi = iS\eta = i\varepsilon\gamma^0\gamma^1\gamma^2\gamma^3$$

qui dépend de l'orientation ε et vérifie :

$$(5.2) \quad \xi\gamma_\alpha + \gamma_\alpha\xi = 0, \quad \tilde{\xi} = \xi, \quad \xi^2 = e.$$

Soit $\mathcal{B}$ (resp. $\overline{\mathcal{B}}$) l'automorphisme de $\mathcal{S}_x$ (resp. $\mathcal{S}'_x$) défini par :

$$\mathcal{B}: \psi \rightarrow \xi\psi, \quad \overline{\mathcal{B}}: \varphi \rightarrow \varphi\xi.$$

On vérifie facilement

$$\mathcal{C}\mathcal{B} = -\mathcal{B}\mathcal{C}, \quad \mathcal{A}\mathcal{B} = -\overline{\mathcal{B}}\mathcal{A}$$

$\mathcal{B}$ admet les valeurs propres ± 1 et $\mathcal{S}_x$ peut être décomposé en somme directe $\mathcal{S}_{x+} \oplus \mathcal{S}_{x-}$ de sous-espaces propres de $\mathcal{B}$ de dimension 2. Un élément de $\mathcal{S}_{x+}$ (resp. $\mathcal{S}_{x-}$) est appelé un spineur positif (resp. négatif). Si $\psi \in \mathcal{S}_x$, $\psi = \psi_+ + \psi_-$ où

$$\psi_+ = \frac{1}{2}(I + \mathcal{B})\psi \in \mathcal{S}_{x+}, \quad \psi_- = \frac{1}{2}(I - \mathcal{B})\psi \in \mathcal{S}_{x-}.$$

Les résultats sont semblables pour $\overline{\mathcal{B}}$. Les applications $\mathcal{C}$ et $\mathcal{A}$ changent le type des spineurs positifs ou négatifs.

6. - Connexion spinorielle.

Une connexion spinorielle est une connexion sur le fibré principal $\mathcal{S}(V_4)$; V_4 est munie naturellement d'une connexion spinorielle: sur un voisinage muni de repères spinoriels et par suite de corepères $\{\theta^a\}$, la connexion riemannienne est définie par la matrice $\omega = (\omega^{\alpha\beta})$ ou (ω^α_β) et la matrice

$$(6.1) \quad \sigma = -\frac{1}{4}\omega^{\alpha\beta}\gamma_\alpha\gamma_\beta = -\frac{1}{4}\omega^\alpha_\beta\gamma^\alpha\gamma^\beta$$

définit notre connexion spinorielle. Ses coefficients sont:

$$\sigma_{b\bar{e}}^a = -\frac{1}{4}c_{\beta\bar{e}}^\alpha(\gamma_\alpha\gamma^\beta)_b^a$$

où les $c_{\beta\bar{e}}^\alpha$ sont les coefficients de rotation de Ricci sur U .

La dérivation covariante dans cette connexion jouit des propriétés suivantes: les dérivées covariantes de γ et de ξ sont nulles. La conjugaison de charge et l'adjonction de Dirac commutent avec la dérivation covariante.

Le tenseur de courbure riemannienne vérifie les relations:

$$(6.2) \quad R_{\alpha\beta,\lambda\mu}\gamma^\beta\gamma^\lambda\gamma^\mu = 2R_{\alpha\beta}\gamma^\beta, \quad R_{\alpha\beta,\lambda\mu}\gamma^\alpha\gamma^\beta\gamma^\lambda\gamma^\mu = -2Re$$

et l'on a l'identité de Ricci pour les 1-spineurs:

$$(6.3) \quad (\nabla_\lambda\nabla_\mu - \nabla_\mu\nabla_\lambda)\psi = -\frac{1}{4}R_{\alpha\beta,\lambda\mu}\gamma^\alpha\gamma^\beta\psi.$$

II - CHAMP DE DIRAC

7. - Opérateurs de Dirac et laplacien.

a) Si ψ est un 1-spineur contrevariant, φ un 1-spineur covariant, introduisons les *opérateurs de Dirac*

$$(7.1) \quad P\psi = \gamma^\alpha\nabla_\alpha\psi, \quad \bar{P}\varphi = -\nabla_\alpha\varphi\gamma^\alpha.$$

On vérifie facilement:

$$(7.2) \quad \mathcal{C}P = P\mathcal{C}, \quad \mathcal{A}P = \bar{P}\mathcal{A}$$

et

$$(7.3) \quad \mathcal{B}P = -P\mathcal{B}, \quad \bar{\mathcal{B}}\bar{P} = -\bar{P}\bar{\mathcal{B}}.$$

Ainsi les opérateurs de Dirac *échangent le type des spineurs positifs et négatifs.*

Le *laplacien* d'un 1-spineur contrevariant (resp. covariant) est l'opérateur $\Delta = P^2$ (resp. $\Delta = \bar{P}^2$). Il est aisé de donner de cet opérateur une expression simple à l'aide de l'identité de Ricci (6.3) et de (6.2). Il vient:

$$(7.4) \quad \Delta = -\nabla^e\nabla_e + \frac{1}{4}R.$$

b) Un *champ de Dirac* classique est défini par un 1-spineur contre-variant satisfaisant

$$(7.5) \quad (P - \mu) \psi = 0 \quad (\mu = \text{const}).$$

L'adjoint de Dirac satisfait alors

$$(7.6) \quad (\bar{P} - \mu) \bar{\psi} = 0.$$

Tout 1-spineur distribution solution de l'équation de Dirac est évidemment solution de l'équation de Klein-Gordon

$$(7.7) \quad (\Delta - \mu^2) \psi = 0.$$

Inversement tout spineur distribution solution de (7.5) se déduit d'une solution de l'équation de Klein-Gordon par l'action de l'opérateur $(P + \mu)$.

8. - Relations de commutation.

a) Désignons par $G^{(\frac{1}{2})\pm}(x, x')$ les deux noyaux élémentaires de l'opérateur $(\Delta - \mu^2)$, c'est-à-dire les deux bi-1-spineurs distributions satisfaisant

$$(8.1) \quad (\Delta_x - \mu^2) G^{(\frac{1}{2})\pm}(x, x') = \Sigma^{(\frac{1}{2})}(x, x')$$

dont les supports sont, pour chaque x' , respectivement dans le futur de x' et dans le passé de x' (*). Sous nos hypothèses, ces deux noyaux élémentaires existent et sont uniques. Plus généralement, tout 1-spineur distribution vérifiant $(\Delta - \mu^2) \psi = 0$ et à support compact soit vers le passé, soit vers le futur est nécessairement nul.

Le propagateur associé à $(\Delta - \mu^2)$ est la différence

$$G^{(\frac{1}{2})} = G^{(\frac{1}{2})-} - G^{(\frac{1}{2})+}.$$

Comme tout propagateur, il est changé en son opposé par changement de q en $-q$.

b) Le propagateur associé aux opérateurs $(P - \mu)$ et $(\bar{P} - \mu)$ est

(*) $\Sigma^{(\frac{1}{2})}$ est le bi-1-spineur de Dirac contrevariant en x , covariant en x' , défini par :

$$\langle \Sigma^{(\frac{1}{2})}(x, x'), \psi(x') \rangle = \psi(x).$$

le noyau:

$$(8.2) \quad S^{(\frac{1}{2})}(x, x') = (P_x + \mu) G^{(\frac{1}{2})}(x, x') = (\bar{P}_{x'} + \mu) G^{(\frac{1}{2})}(x, x') .$$

On voit à l'aide du théorème d'unicité qu'il vérifie:

$$(8.3) \quad \bar{S}^{(\frac{1}{2})}(x', x) = -S^{(\frac{1}{2})}(x, x') , \quad \overset{\circ}{S}^{(\frac{1}{2})}(x, x') = S^{(\frac{1}{2})}(x, x') .$$

c) Si ψ est à composantes opératoriellles, l'anticommutateur correspondant est donné par:

$$(8.4) \quad [\psi(x), \bar{\psi}(x')]_+ = \frac{1}{i} S^{(\frac{1}{2})}(x, x') E$$

où le second membre a son support dans l'émission de x' pour chaque x' , vérifie les équations de Dirac et est compatible d'après (8.2) avec son interprétation comme anticommutateur.

9. - Lagrangien, tenseur canonique et vecteur-courant.

a) Si φ est un 1-spineur opératoriel covariant, ψ un 1-spineur opératoriel contrevariant, nous posons dans la suite

$$(\varphi\psi)_- = \frac{1}{2}(\varphi_a \psi^a - \psi^a \varphi_a) .$$

Aux équations de Dirac est associé le lagrangien opératoriel:

$$(9.1) \quad L(\psi) = \frac{1}{2}(\bar{\psi}\gamma^\alpha \nabla_\alpha \psi - \nabla_\alpha \bar{\psi} \gamma^\alpha \psi)_- - \mu(\bar{\psi}\psi)_- .$$

On en déduit le tenseur canonique

$$(9.2) \quad T^{\alpha\beta}(\psi) = \frac{1}{2}(\nabla^\alpha \bar{\psi} \gamma^\beta \psi - \bar{\psi} \gamma^\beta \nabla^\alpha \psi)_-$$

qui, en vertu des équations de champ, vérifie les relations:

$$\nabla_\beta T^\beta_\alpha(\psi) = 0 , \quad \nabla_\alpha T^\alpha_\beta(\psi) = 0 .$$

Le tenseur symétrique

$$\theta^{\alpha\beta}(\psi) = \frac{1}{2}\{T^{\alpha\beta}(\psi) + T^{\beta\alpha}(\psi)\}$$

est conservatif et ne diffère de $T^{\alpha\beta}(\psi)$ que par une divergence covariante.

Le vecteur courant de la théorie est donné par:

$$(9.3) \quad J^\alpha(\psi) = i(\bar{\psi}\gamma^\alpha\psi)_-$$

et vérifie, en vertu des équations de champ:

$$\nabla_\alpha J^\alpha(\psi) = 0.$$

b) On vérifie immédiatement à l'aide de (7.2) et (8.2) que, *par la conjugaison de charge* $\mathcal{C}$, les équations de champ et les relations de commutation sont invariantes. De plus pour $\mathcal{C}$:

$$L(\psi^c) = L(\psi), \quad T(\psi^c) = T(\psi), \quad J(\psi^c) = -J(\psi).$$

Cette situation est décrite en disant que la théorie est $\mathcal{C}$ -invariante. On appelle ici *transformation* $\mathcal{P}$ le changement *d'orientation spatiale* $\sigma \rightarrow -\sigma$. Tous les éléments introduits étant manifestement invariants par $\mathcal{P}$, la théorie est $\mathcal{P}$ -invariante.

On appelle *transformation* $\mathcal{T}$ le changement *d'orientation temporelle* $\varrho \rightarrow -\varrho$ complété par le changement $i \rightarrow -i$ sur les grandeurs non opératoriels. Il est clair que $\bar{\psi} \rightarrow \bar{\psi}$ par $\mathcal{T}$. Les équations de champ et les relations de commutation sont invariantes par $\mathcal{T}$. De plus pour $\mathcal{T}$:

$$\mathcal{T}: \quad L(\psi) \rightarrow L(\psi), \quad T(\psi) \rightarrow T(\psi), \quad J(\psi) \rightarrow -J(\psi).$$

Il y a $\mathcal{T}$ -invariance de la théorie.

Il est clair que le produit $\mathcal{P}\mathcal{T}\mathcal{C}$ des transformations laisse invariant chacune des quantités et chacune des équations envisagées.

III - CHAMP NEUTRINIQUE

10. - Champ neutrinique.

Un champ neutrinique est un champ de Dirac pour lequel $\mu = 0$. Un tel champ quantique est donc représenté par un 1-spineur opératoriel vérifiant l'équation de champ

$$(10.1) \quad P\psi = 0.$$

Décomposons ψ selon les types positif ou négatif

$$(10.2) \quad \psi = \psi_+ + \psi_-$$

$\mathcal{C}$ permutant les types, on a :

$$(10.3) \quad \mathcal{C}\psi = \mathcal{C}\psi_+ + \mathcal{C}\psi_-$$

où $\mathcal{C}\psi_+$ est négatif et $\mathcal{C}\psi_-$ est positif. Une remarque analogue est valable pour $\mathcal{A}$.

D'après les propriétés de P , l'éq. (10.1) se décompose selon les deux équations de Dirac :

$$(10.4) \quad P\psi_+ = 0, \quad P\psi_- = 0.$$

Le champ neutrinique ψ étant ainsi décomposé en deux champs ψ_+ et ψ_- vérifiant les équations de Dirac et respectivement de type positif et négatif, nous sommes amenés à étudier la décomposition selon les types de l'anticommutateur correspondant.

11. - Anticommutateurs des champs de neutrinos.

a) Des propriétés du bi-1-spineur de Dirac, il résulte

$$\Sigma^{(\frac{1}{2})}(x^+, x'^-) = \Sigma^{(\frac{1}{2})}(x^-, x'^+) = 0$$

où les symboles $+$ ou $-$ représentent les projections selon les types. Du théorème d'unicité on déduit

$$G^{(\frac{1}{2})\pm}(x^+, x'^-) = G^{(\frac{1}{2})\pm}(x^-, x'^+) = 0$$

et par suite

$$G^{(\frac{1}{2})}(x^+, x'^-) = G^{(\frac{1}{2})}(x^-, x'^+) = 0.$$

P changeant les types, on déduit

$$(11.1) \quad S^{(\frac{1}{2})}(x^+, x'^+) = S^{(\frac{1}{2})}(x^-, x'^-) = 0.$$

b) De l'anticommutateur (8.3), on déduit par décomposition selon les types

$$(11.2) \quad [\psi_+(x), \bar{\psi}_+(x')]_+ = \frac{1}{i} S^{(\frac{1}{2})}(x^+, x'^-) [\psi_-(x), \bar{\psi}_-(x')]_+ = \frac{1}{i} S^{(\frac{1}{2})}(x^-, x'^+)$$

ainsi que

$$(11.3) \quad [\psi_+(x), \bar{\psi}_-(x')]_+ = 0, \quad [\psi_-(x), \bar{\psi}_+(x')]_+ = 0.$$

Nous obtenons ainsi deux champs indépendants de types positif ou négatif admettant les anticommutateurs (11.2). On en déduit

$$(11.4) \quad [\mathcal{C}_x \psi_+(x), \mathcal{C}_{x'} \psi_+(x')]_+ = \\ = \frac{1}{i} S^{(\frac{1}{2})}(x^-, x'^+), [\mathcal{C}_x \psi_-(x), \mathcal{C}_{x'} \psi_-(x')]_+ = \frac{1}{i} S^{(\frac{1}{2})}(x^+, x'^-)$$

Nous nous trouvons en présence de 4 champs, se correspondant deux à deux par conjugaison de charge et qui décrivent des particules que nous notons:

$$\begin{array}{ll} \psi_+ : \nu & \mathcal{C}\psi_+ = \psi_-^c : \bar{\nu} \\ \psi_- : \bar{\nu}' & \mathcal{C}\psi_- = \psi_+^c : \nu' \end{array}$$

Les particules ν et ν' sont dites des *neutrinos*, les particules $\bar{\nu}$ et $\bar{\nu}'$ des *antineutrinos*.

12. - Lagrangien, tenseur canonique et courant du neutrino.

a) Considérons un champ neutrinique ψ et sa décomposition $\psi_+ + \psi_-$. Aux équations de Dirac relatives à ψ_+ est associé le lagrangien du neutrino:

$$(12.1) \quad L_+(\psi) = L(\psi_+) = \frac{1}{2}(L(\psi) + M(\psi))$$

avec

$$M(\psi) = -\frac{1}{2}(\bar{\psi}\xi\gamma^\alpha\nabla_\alpha\psi + \nabla_\alpha\bar{\psi}\gamma^\alpha\xi\psi)_-$$

De ce lagrangien dérive le tenseur canonique

$$(12.2) \quad T_+^{\alpha\beta}(\psi) = T^{\alpha\beta}(\psi_+) = \frac{1}{2}(T^{\alpha\beta}(\psi) + U^{\alpha\beta}(\psi))$$

avec

$$U(\psi) = \frac{1}{2}(\nabla^\alpha\bar{\psi}\gamma^\beta\xi\psi + \bar{\psi}\xi\gamma^\beta\nabla^\alpha\psi)_-$$

Le vecteur-courant s'écrit

$$(12.3) \quad J_+^\alpha(\psi) = J^\alpha(\psi_+) = \frac{1}{2}(J^\alpha(\psi) + K^\alpha(\psi))$$

$$K^\alpha(\psi) = i(\bar{\psi}\gamma^\alpha\xi\psi)_-$$

b) Il n'y a pas $\mathcal{C}$ invariance de la théorie. En particulier:

$$L_+(\psi^c) = \frac{1}{2}(L(\psi) - M(\psi)), \quad T^{\alpha\beta}(\psi^c) = \frac{1}{2}(T^{\alpha\beta}(\psi) - U^{\alpha\beta}(\psi)), \\ J_+^\alpha(\psi^c) = -\frac{1}{2}(J^\alpha(\psi) - K^\alpha(\psi)).$$

Par la transformation $\mathcal{P}$: $\sigma \rightarrow -\sigma$, ξ se transforme selon $\xi \rightarrow -\xi$. Les équations de champ et les relations de commutation sont modifiées

$$L_+(\psi) \rightarrow \frac{1}{2}(L(\psi) - M(\psi)), \quad T_+^{\alpha\beta}(\psi) \rightarrow \frac{1}{2}(T^{\alpha\beta}(\psi) - U^{\alpha\beta}(\psi)), \\ J_+^\alpha(\psi) = \frac{1}{2}(J^\alpha(\psi) - K^\alpha(\psi)).$$

Il n'y a pas $\mathcal{P}$ -invariance (non conservation de la parité). En ce qui concerne la transformation $\mathcal{T}$: $\varrho \rightarrow \varrho$, $i \rightarrow -i$, le 2-spineur ξ n'est pas modifié et $\bar{\psi} \rightarrow \bar{\psi}$. Les équations de champ et relations de commutation demeurent invariantes. Lagrangien et tenseur canonique ne sont pas modifiés, tandis que le vecteur-courant est changé en son opposé.

Il y a $\mathcal{T}$ -invariance. Il y a aussi invariance par le produit $\mathcal{PTC}$ des transformation.

IV - TRANSFORMATION $\mathcal{PTC}$

13. - Champ du méson vectoriel.

a) Un champ mésonique vectoriel est décrit par une 1-forme α à valeurs opératoriels, vérifiant l'équation de champ

$$(13.1) \quad \delta d\alpha = \mu^2 \alpha \quad (\mu = \text{const} \neq 0)$$

où d est l'opérateur de différentiation extérieure et δ celui de codifférentiation. L'équation (13.1) est équivalente au système

$$(13.2) \quad (\Delta - \mu^2)\alpha = 0$$

et

$$(13.3) \quad \delta\alpha = 0$$

où $\Delta = d\delta + \delta d$ est le « laplacien » sur les formes. Le commutateur correspondant est

$$(13.4) \quad [\alpha(x), \alpha^\#(x')]_- = \frac{1}{i} \left\{ G^{(1)}(x, x') - \frac{1}{\mu^2} d_x \bar{d}_{x'} G^{(0)}(x, x') \right\} E$$

où $G^{(0)}$, $G^{(1)}$ sont les propagateurs scalaires et vectoriels relatifs à $(\Delta - \mu^2)$. L'équation (13.1) dérive du lagrangien

$$(13.5) \quad L(\alpha) = \frac{1}{2}(\bar{d}\alpha_{\lambda\mu} d\alpha^{\# \lambda\mu})_+ - \frac{\mu^2}{2}(\alpha_\lambda \alpha^{\# \lambda})_+.$$

Il lui correspond le tenseur canonique

$$(13.6) \quad T^{\alpha\beta}(\alpha) = \frac{1}{2}(\bar{d}\alpha_\mu^\alpha d\alpha^{\# \beta\mu} + \bar{d}\alpha^{\# \alpha}_\mu d\alpha^{\beta\mu})_+ - \frac{\mu^2}{2}(\alpha^\alpha \alpha^{\# \beta} + \alpha^{\# \alpha} \alpha^\beta)_+ - gL^{\alpha\beta}(\alpha)$$

et le vecteur-courant

$$(13.7) \quad J^\alpha(\alpha) = \frac{i}{2}(\alpha_e d\alpha^{\# \alpha e} - \alpha_e^{\#} d\alpha^{\alpha e})_+$$

c) A la 1-forme α , associons le 2-spineur

$$(13.8) \quad \psi = i\gamma^e \alpha \varrho.$$

Par $\mathcal{C}$: $\psi \rightarrow \psi^c = \psi^{\#}$ et $\alpha \rightarrow \alpha^c = -\alpha^{\#}$. De même par $\mathcal{T}$: $\psi \rightarrow \psi^{\mathcal{T}} = -\psi$ et $\alpha \rightarrow \alpha^{\mathcal{T}} = -\alpha$.

On vérifie immédiatement qu'il y a aux sens précédents, $\mathcal{C}$ -invariance $\mathcal{P}$ -invariance et $\mathcal{T}$ -invariance.

14. - Les transformations H et K .

a) Désignons par H , ou conjugaison hermitienne, la transformation définie par la substitution à chaque champ de son conjugué de charge, complété par le changement $i \rightarrow -i$ et par l'inversion de l'ordre des opérateurs. Pour les spins 0, $\frac{1}{2}$ et 1 on a par exemple:

$$u \rightarrow u^{\#}, \quad \psi \rightarrow \psi^{\#} \text{ et } \bar{\psi} \rightarrow -\bar{\psi}^{\#}, \quad \alpha \rightarrow \alpha^{\#}.$$

Pour toutes les théories quantiques de champ envisagées, H est un isomorphisme de l'algèbre des opérateurs (les relations de commutation sont invariantes) laissant invariants le lagrangien, le tenseur canonique, le vecteur courant.

b) Désignons par K , ou symétrie totale, la transformation définie par $\varrho \rightarrow -\varrho$, $\sigma \rightarrow -\sigma$ complétée par l'inversion de l'ordre des opérateurs; K est encore un isomorphisme de l'algèbre des opérateurs laissant invariants le lagrangien, le tenseur canonique, le vecteur courant.

Dans une telle théorie, tous les éléments et équations introduits sont invariants par le produit KH . Or de la définition de H et K , on déduit immédiatement:

$$KH = \mathcal{PTC}.$$

Aussi toute théorie quantique de champ invariante par conjugaison hermitienne et par symétrie totale est $\mathcal{PTC}$ -invariante.

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INTERVENTI E DISCUSSIONI

— J. EHLERS:

I should like to ask two questions:

1) It is possible to extend the concept of covering group to disconnected group; in other words is it true even for a disconnected group that a covering group is determined uniquely up to an isomorphism?

2) In the special relativistic formulation of quantum fields by Wightman *et al.* essential use is made of Fourier transforms in constructing the Fock space. In a curved background space-time Fourier transforms are not available. Is it nevertheless possible to construct a Fock-space, creation operators etc. in a precise manner?

— A. LICHNEROWICZ:

Je répondrait à vos deux questions:

1) J'ai parlé ici un peu sommairement du groupe revêtement d'ordre 2 du groupe de Lorentz homogène complet. Je donne un traitement complet et détaillé de cette question dans *Annales Institut Henri Poincaré* (1964).

2) La transformation de Fourier est un excellent instrument dans le cas plat. Mais elle n'est pas indispensable à la théorie. J'ai donné des indications concernant l'introduction des opérateurs création-annihilation et la construction d'un espace de Fock, dans mon cours de l'Ecole de Houches (1963). En ce qui concerne les opérateurs de création, on peut partir d'une définition de G (propagateur généralisant Δ) et G_1 (noyau symétrique généralisant le Δ_1 du cas plat) reliés par une relation intégrale fondamentale dont toute la théorie dérive.

— S. DESER:

Since there is in fact close correspondence between your $\mathcal{P}$, $\mathcal{C}$, $\mathcal{T}$ operations and the standard (Jost, etc.) ones, and since the usual results yield $\mathcal{TCP}$ invariance for arbitrary interaction from the assumptions of weak local commutativity, positive energy and proper Lorentz invariance, is there any difficulty in extending your results to coupling? Of course you work on the Lagrangian, rather than in terms of analyticity of spectral functions, but that is presumably no major obstacle.

— A. LICHNEROWICZ:

I work on this problem and I think that there is no real obstacle.

— O. COSTA DE BEAUREGARD:

Je pense que vos définitions manifestement intrinsèques, constituent un progrès considérable sur les précédentes définitions, toutes assez ambiguës. En particulier je suis frappé du caractère manifeste de l'existence de deux systèmes neutrino-antineutrino.

— A. LICHNEROWICZ:

Je suis heureux de votre appréciation.

— O. COSTA DE BEAUREGARD:

Je pense que l'éventualité physique d'un recouplement des géodésiques du genre temps et d'une refermeture des cônes isotropes n'a rien de fondamentalement absurde. Dès lors que le champ gravitationnel encourbe les trajectoires de points matériels (y compris celles des photons) il se comporte comme un « milieu optique »; éventuellement comme un « système optique » à focalisation plus au moins parfaite. Il me semble donc que, physiquement parlant, il ne faut pas exclure cette éventualité.

— A. LICHNEROWICZ:

C'est ce que je cherche à faire en remplaçant des hypothèses trop fortes sur le comportement du cône caractéristique (hypothèses souvent plus ou moins implicites) par l'hypothèse de globale hyperbolicité, beaucoup plus faible, mais qui assure l'existence et l'unicité des propagateurs.

— J. GÉHÉNIAT:

Vous attribuez la même hélicité aux deux neutrinos ν et ν' . D'autre part, on donne habituellement à l'électron négatif e^- le nombre leptonique $L = +1$, et l'on appelle antineutrino $\bar{\nu}$ la particule neutre émise dans la désintégration du neutron. Ce ν a donc l'hélicité $H = +1$ (expérimental). Avec cela et la loi de conservation du nombre leptonique, les leptons se placent comme suit dans les Tableaux I et II, si l'on n'accepte qu'une particule par case.

TABLEAU I.

$H \backslash L$	+	—
+	$\bar{\nu}'$	$\bar{\nu}$
—	ν	ν'

TABLEAU II.

$Q \backslash L$	+	—
+	μ^+	e^+
—	e^-	μ^-

Bien sûr, beaucoup d'autres conventions sont possibles. Remarquer que la dernière convention (une particule par case) fixe l'hélicité du neutrino (ici $\bar{\nu}$) qui accompagne le muon négatif, par exemple, dans la désintégration du pion négatif.

— A. LICHNEROWICZ:

Je suis complètement d'accord.

— A. H. TAUB:

In the definition of the Dirac operator given the γ^a and μ were taken to be real. A suitable Dirac operator may be defined by taking μ to be pure imaginary. If this is done is the resulting theory isomorphic to that given here? Is the same true when γ^a is replaced by $\xi\gamma^a$?

— A. LICHNEROWICZ:

Je pense comme vous qu'un bon exposé, du point de vue mathématique, devrait être intrinsèque et n'introduire que des isomorphismes et antiisomorphismes, indépendamment de toute base. En ce qui concerne des travaux concernant les spineurs en géométrie différentielle globale, j'ai moi-même fait un tel exposé.

Bien entendu le choix fait ici de matrices de Dirac *réelles* est parfaitement inessential. Il est seulement commode pour distinguer à simple inspection les invariances et permet d'éviter quelques calculs. C'est une commodité d'exposition.

— A. MERCIER:

1) La nécessité relevée par l'orateur de remplacer l'idée naïve du physicien des conditions du théorème $\mathcal{CPT}$ (intervention « du passé lointain avec l'avenir lointain » etc.) par des conditions formulées correctement dans le formalisme covariant rappelle celle, élémentaire, concernant le théorème de réversibilité de la mécanique rationnelle. En effet, on voit souvent exprimer la réversibilité de la mécanique « par rapport à un changement de signe du temps t », ce qui est inexact, alors qu'il faut imaginer que l'on remplace dt par $-dt$ et (par exemple) les dp_k par $-dp_k$, c'est-à-dire opérer avec les différentielles, ce qui est clair car le mouvement, à part des cas exceptionnels, n'a pas lieu dans un espace « plat » (p. ex. la surface d'énergie). La généralisation de la démonstration de $\mathcal{CPT}$ est donc une nécessité, en ce qu'elle montre au juste en quoi consistent les conditions, ce qu'on ne voit pas dans une métrique pseudo-euclidienne.

2) Historiquement, on ne pouvait savoir d'emblée qu'il devrait y avoir deux neutrinos et leurs antineutrinos, parce qu'aucun mathématicien n'avait démontré la décomposition neutrinique du spineur, tandis que p. ex. Dirac avait pu annoncer l'existence de ce qui s'est appelé plus tard l'antiparticule, puisque sa théorie la contenait.

3) M. Lichnerowicz a insisté sur l'hypothèse qu'il a faite que les géodésiques isotropes ne se recoupent pas et qui trouve son expression mathématique à l'alinéa (2.b) de son article (hyperbolicité globale de V_4). Je lui poserai

alors la question de savoir si le seul danger que l'abandon de cette hypothèse présente résiderait dans l'application de ses idées à la théorie de la matrice S .

— A. LICHNEROWICZ:

Vos remarques sont extrêmement intéressantes.

1) Dans la plupart des considérations classiques relatives à l'espace temps plat se mélangent trois choses qu'il convient de distinguer: la symétrie par rapport à un 3-plan orienté spatialement, la définition du passé et du futur, l'introduction de repères dont le vecteur temporel est vers le passé ou le futur. Le premier point de vue n'a pas de sens en relativité générale, le second correspond à l'introduction de cet objet géométrique que je nomme l'orientation temporelle φ , le troisième n'a pas d'intérêt si l'on introduit systématiquement des objets géométriques (comme $\bar{\varphi}$) invariants dans le groupe de Lorentz complet.

2) Je possédais depuis longtemps cette théorie à 2 neutrinos (et 2 anti-neutrinos), mais je n'osais pas le rendre public en l'absence de toute expérience. Je l'ai exposée quelques semaines après l'apparition des deux neutrinos expérimentaux.

3) Mon hypothèse assure l'existence de solutions élémentaires ou « fonctions de Green ».

— S. DESER:

With respect to Professor Lichnerowicz's important point, it is only the integrated energy expression which is independent of the spatial vierbein orientation once a choice of time gauge is made. On the other hand, the energy density remains highly dependent on the spatial orientation even when the time axis is decided; this dependence must be regarded, as Lichnerowicz points out, as a difficulty of the local quantities.

— M. A. TONNELAT:

Le formalism est-il modifié en passant aux spins supérieurs ($\frac{3}{2}$, $\frac{5}{2}$, ...)? En particulier peut-on définir les noyaux élémentaires et les relations de commutation correspondantes?

— A. LICHNEROWICZ:

Le formalisme introduit ici est valable en spin quelconque. J'ai donné ailleurs explicitement le cas de spin $\frac{3}{2}$ (formalisme de Rarita-Schwinger). Il n'y a pas de difficultés majeures (voir *Bull. Soc. Math. France* (1964)).

— S. BONAZZOLA:

Je voudrais vous poser deux questions:

1) Quelle signification physique à votre avis peut-on attacher à la condition d'hypérbolité globale, dont vous avez parlé au début de votre exposé?

2) Pourquoi à votre avis le problème des particules élémentaires peut-il être résolu seulement dans un espace courbe?

— A. LICHNEROWICZ:

1) La signification est la suivante: si deux points de l'espace temps sont reliés temporellement, de cette suite infinie de chemins temporels reliant ces

deux points, on peut extraire une suite de chemins convergeant vers un chemin temporel. Dans le cas usuel, cela exclut les cas où le cône élémentaire tendrait à s'aplatir selon un 3-plan ou à s'affiner selon une direction. Inversement on peut démontrer qu'il y a hyperbolicité globale si l'angle du cône reste borné, des deux côtés, loin de cas de dégénérescence.

2) Ce n'est pas exact: le problème des particules élémentaires ne nécessite pas un espace temps courbe. Mais il y a un avantage à raisonner sur un espace temps courbe. Les renonces sont moins nombreuses et vous êtes obligés de distinguer l'essentiel. On pourrait dire qu'étant plus pauvre en structures que la relativité restreinte, la relativité générale vous oblige à élaborer des *meilleures théories*.

Les théorèmes de conservation et l'idée d'une théorie unitaire.

A. MERCIER

Université de Berne - Berne

È un grand'onore che mi viene fatto di dover parlare in questa augusta assemblea riunita per commemorare la nascita di un uomo che ci sembra oggi come il primo tra i primi « moderni ». Quando ci si pensa, ci si accorge che non siamo noi « moderni », oggi. La nostra scienza si radica in quattro secoli di tradizione e di tradizioni e siamo per quella ragione imprigionati in un sistema di pensiero che per una grande parte ci impedisce di trovare in una libertà totale la *fisica nuova* che ci fa bisogno. È vero che la generazione precedente ha visto sorgere due giganti del pensiero esatto che sono i primi tra i nostri maestri. Ma se Einstein ha rinnovato completamente l'idea galileiana della relatività e la concezione newtoniana della gravitazione, ci appare, nella nostra generazione, come un classico o un poco come l'ultimo rappresentante di una maniera di pensare che definisce un'epoca della storia della cultura occidentale. E se Bohr ci ha liberato da una credenza in un determinismo ch'era diventato un mito e mostrato che la fisica quantica non contraddice la biologia, la psicologia e le altre imprese dello spirito umano, non ci ha legato la super-teoria che cerchiamo ancora tutti noi.

Come l'ha raccolto l'istorico americano delle scienze de Solla Price, la nostra generazione comprende tanti e più sapienti di quanti ci sono stati da Galileo alla generazione precedente. Ciò mostra che la quantità non è sufficiente per assicurare l'esistenza dei giganti. Il genio è una apparizione misteriosa.

E adesso mi rivolgerò verso il tema che gli organizzatori del simposio hanno scelto e le questioni fondamentali che ci sono state poste dai colleghi italiani.

Ci sono due modi possibili di attaccare il problema: o di trattare una questione specifica più o meno nuova e valutare un fenomeno speciale alla luce della teoria che possediamo, o considerare il problema nella sua

totalità per estrarne ciò che ci sembra caduco e farne una critica generale, tentando di aprire finestre verso ciò che potrebbe essere l'origine di una spiegazione realmente nuova.

Ho scelto la seconda via, colla coscienza di fare niente di più che opera speculativa. Ma siamo insieme per apprendere.

Dans ce travail, nous aimerions présenter quelques réflexions d'ordre élémentaire sur les idées de conservation en physique et montrer à ce propos comment évolue l'aspect épistémologique des théories physiques.

I. — Les réflexions philosophiques sur la science n'ont guère été fructueuses que lorsqu'elles se sont appuyées sur des résultats scientifiques concrets, mais en même temps elles ont toujours été liées à certaines *vues* nouvelles de la science, alors que ces vues ont toujours quelque chose d'éphémère.

Ainsi, au 19^e siècle, le développement de l'énergétique fit une impression considérable sur la pensée philosophique. La notion de matière subit une évolution et l'antique idée de conservation de la substance fit peu à peu place à celle de conservation de l'énergie. On ne remarqua pas que la conservation de l'énergie est encore moins une conservation véritable que celle de la quantité de mouvement, à cause du sens indéfini de l'énergie potentielle, et l'on fut complètement subjugué par le succès de l'introduction de nouvelles « formes » d'énergie: calorifique, électromagnétique, et finalement massique. A l'introduction de chaque nouvelle forme, on sauvait la conservation du danger de n'être pas valable dans certains cas et on enrichissait en même temps le vocabulaire de la physique d'une notion spécifique importante supplémentaire.

La relativité restreinte sembla montrer à la fois que la dite conservation de l'énergie n'est pas une conservation au sens strict mais seulement la projection parallèle à l'axe des temps d'une relation de conservation quadrimensionnelle, et que cette relation est un renforcement de l'idée de conservation en ce que l'énergie en flux passe à la quantité de mouvement et inversement — comme en une nouvelle transformation —, en ce que la conservation du quadrivecteur énergie-momentum semblait compléter admirablement l'idée originale et en ce que la densité d'énergie et celle de la quantité de mouvement s'adjoignait à des tensions du type de Maxwell pour former une densité tensorielle $\mathcal{T}$. Mais à ce point, on pouvait difficilement s'apercevoir que l'équation correspondante $\text{Div } \mathcal{T} = 0$ n'est valable en grand et non dans un petit entourage seulement qu'en vertu du caractère pseudo-euclidien de la métrique imposée à la variété d'espace-temps dans laquelle on décrivait ces phénomènes, alors que lorsqu'on a voulu la généraliser à une métrique riemannienne, le flux ne compensa plus simplement la variation de la densité dans le temps.

On a alors discuté tout au long le sens d'une forme non-invariante

des relations de conservation, en introduisant un pseudo-tenseur t des tensions du champ de gravitation, un tenseur de superénergie ou un artifice de ce genre.

Selon certains auteurs, les notions d'énergie et de quantité de mouvement n'ont que peu d'importance en relativité générale (cf. P. G. Bergmann: *Introduction to the Theory of Relativity*, N. Y. 1947, footnote 5, p. 193); une opinion pareille repose, semble-t-il, sur la constatation pure et simple de l'impossibilité de définir une conservation stricte. D'autres auteurs (W. Scherrer en particulier) estiment que l'un des plus graves défauts, peut-être même le seul défaut de la théorie d'Einstein, est de ne pas permettre la démonstration d'un théorème de conservation stricte de l'énergie; cette autre opinion repose apparemment sur le désir de respecter dans son intégrité l'antique idée de conservation de la substance par l'intermédiaire de l'énergie.

L'introduction d'artifices peut être due à diverses tendances; les uns cherchent un compromis par une interprétation acceptable. On imagine une pression travaillant à l'expansion de l'univers, sans qu'il y ait cependant ni piston pour l'exercer, ni membrane dia-énergétique pour y résister à la limite de l'univers, d'où la nécessité de localiser l'énergie échappée sous forme potentielle dans les masses qui s'éloignent.

En dehors des masses, où il n'y a que du champ gravitationnel mais pas de densité matérielle, des ondes gravitationnelles emmènent avec elles l'énergie et la quantité de mouvement, bien que la densité de ces dernières y soit partout et constamment nulle, ce qui apparaît paradoxal, et on se demande ce que ces ondes peuvent transporter réellement. Il s'en est suivi qu'il a fallu chercher en quoi consiste le *vrai* champ de gravitation.

Ces péripéties paraissent montrer deux choses. L'une est que la place singulière accordée autrefois à la notion d'énergie est exagérée, l'autre que ce n'est pas à propos de l'énergie (ni de la quantité de mouvement) qu'un principe de conservation s'énonce strictement et que par conséquent l'idée de substance indestructible, si elle a un sens, doit être rattachée à autre chose.

2. — Classiquement, c'est-à-dire dans la représentation newtonienne tridimensionnelle, on peut dire que

la densité μ et le flux $\sigma = \mu v$ de la masse,

la densité ϱ et le flux $s = \varrho v$ de la charge électrique

et la densité ε et le flux $\gamma = \varepsilon v$ de l'énergie

conduisent à une équation de continuité, à savoir

$$\begin{pmatrix} a \\ 1 \cdot b \\ c \end{pmatrix} \quad \frac{\partial}{\partial t} \begin{pmatrix} \mu \\ \varrho \\ \varepsilon \end{pmatrix} + \nabla \cdot \begin{pmatrix} \sigma \\ s \\ \gamma \end{pmatrix} = 0.$$

La forme en est identique, et l'on a l'impression qu'on a affaire dans les trois cas à une véritable conservation selon le théorème de Gauss

$$(1') \quad \frac{d}{dt} \int \begin{pmatrix} \mu \\ \varrho \\ \varepsilon \end{pmatrix} d\tau = 0.$$

Cette conclusion est liée aux transformations de Galilée.

Mais, si elle était vraiment valable, pourquoi y aurait-il une différence physique entre μ , ϱ et ε ? Il semble difficile qu'on puisse distinguer par l'expérience ces trois conservations de forme identique. Il ne devrait y avoir qu'une seule conservation galiléenne.

C'est d'ailleurs ce qu'on postule en théorie quantique ordinaire, où l'on démontre une équation de conservation

$$(2) \quad \frac{d}{dt} \int |\psi|^2 d\tau = 0$$

en vertu des propriétés d'hermiticité de l'opérateur hamiltonien et du fait que, seul parmi toutes les observables, cet opérateur donne lieu à une équation fondamentale d'évolution des systèmes.

Dans le cas d'une particule unique de charge e et de masse m , on définit $\begin{pmatrix} \mu \\ \varrho \end{pmatrix} \equiv \begin{pmatrix} m \\ e \end{pmatrix} |\psi|^2$ et l'on a (1.a) et (1.b) conséquence de (2). Cependant, il ne vient à l'idée de personne d'identifier « en substance » la masse et la charge. On s'en garderait bien, à cause d'un préjugé dicté par la relativité selon lequel e est un invariant de Lorentz, tandis que m ne l'est pas. Ou alors, il faut s'en référer à l'équation de Dirac; L. de Broglie (cf. *L'électron magnétique*) a bien montré que dans ce cas une identification de e et de m est impossible. Or, un raisonnement classique, c'est-à-dire préquantique et prérelativiste, ne devrait pas se laisser influencer par des considérations pareilles. Le fait que c'est toujours e/m qui apparaît dans les relations dynamiques décrivant des expériences du type de celle de J. J. Thomson aurait pu suggérer qu'il n'y a pas de différence entre la charge et la masse.

De fait, (1.b) s'établit à partir des deux équations inhomogènes de

Maxwell qui ne sont pas de la nature d'une équation dynamique, mais de celle d'une relation de source à champ. Elles ne contiennent donc rien qui ressemble à un hamiltonien.

(1.c), en revanche, découle d'une équation dynamique, et, mise sous la forme (1'), elle entraîne une première intégrale du mouvement, ce qui ne peut pas être dit au même sens de (1, a ou b). (1.c) s'obtient, il est vrai, en électrodynamique, à partir des équations de Maxwell, mais son établissement requiert que l'on s'appuie non seulement sur une paire d'équations dont l'une est homogène et l'autre inhomogène, mais aussi sur la définition de la force de Lorentz ou mieux, puisque (1.c) ne doit pas contenir de densité de puissance développée, sur une interprétation forcée où il n'est pas établi d'emblée que γ est égal à ev .

Cela déjà nous montre que (1.c) n'est pas une véritable conservation.

En outre, dans le cadre de la théorie quantique ordinaire des particules, il ne vient à personne l'idée d'attribuer à la particule une densité d'énergie proportionnelle à $|\psi|^2$; en effet, de quelle énergie s'agirait-il? Dans ce cas, la démonstration de la conservation de l'énergie nécessite l'hypothèse supplémentaire $\partial H/\partial t = 0$.

Au fond, du point de vue physique et en mécanique rationnelle déjà la conservation de l'énergie est un peu un mythe, car les systèmes conservatifs sont une idéalisation et il n'y a pas, dans la nature, de système rigoureusement conservatif, de sorte qu'il faut avoir recours à la thermodynamique, qui n'est qu'un sauvetage et non une démonstration de la conservation. N'importe quel corps céleste est en présence, si éloignée soit elle, d'au moins deux autres corps, et, n'étant pas ponctuel, soumis conséquemment à des marées qui développent de la chaleur. Quant aux particules, elles dissipent vraisemblablement toutes d'une façon ou d'une autre leur énergie comme le fait déjà une particule chargée qui ne se trouve jamais en mouvement rigoureusement uniforme et rectiligne puisqu'il y en a d'autres pour les accélérer; en d'autres termes, il n'y a aucune raison de croire qu'il ait des particules ne dissipant pas leur énergie lorsqu'elles sont, du point de vue classique, accélérées, donc en couplage avec d'autres qui en sont issues et leur soutirent une portion de leur énergie lorsqu'elles font apparition.

L'idée d'une conservation de l'énergie, issue il est vrai d'un théorème de mécanique rationnelle mais généralisée psychologiquement à cause de celle de l'indestructibilité de la substance, ne se rattache en vérité pas logiquement à cette indestructibilité. C'est seulement le cas de la charge, ce que confirme la relativité restreinte.

On peut se demander d'ailleurs si une vraie loi de conservation ne s'applique pas seulement à des grandeurs physiques de nature strictement scalaire et algébrique, c'est-à-dire pour lesquelles il y a des quantités positives et négatives capables de se compenser, qu'elles soient discrètes

dans le cas de la matière discontinue, ou que ce soit continûment. Ce n'est pas le cas de l'énergie, à moins qu'on parvienne à faire une physique où il n'y ait plus d'énergie potentielle qui n'a pas de valeur absolue définie mais seulement une énergie d'inertie des deux signes en vertu d'une formule telle que

$$(3) \quad E^2 = m_0^2 c^4 + c^2 p^2$$

et même en la restreignant aux transformations de Galilée!

Or, même l'éq. (3) est peut-être fausse, ou du moins mal écrite. Le formalisme de la relativité restreinte auquel elle se rattache est un formalisme canonique homogène, c'est-à-dire que x^4 y figure comme $(f+1)$ -ème coordonnée, f étant le nombre de degrés de liberté au sens du formalisme inhomogène habituel. Alors les équations canoniques s'étendent aux $(f+1)$ degrés de liberté homogènes:

$$(4) \quad \frac{d\Phi}{d\tau} = [\Phi, \mathcal{H}] , \quad \Phi = \Phi(q_1 \dots q_{f+1}, p_1 \dots p_{f+1}) ,$$

où τ est un paramètre de référence et $\mathcal{H}$ une hamiltonienne homogène, à la condition que cette dernière s'annule:

$$(5) \quad \mathcal{H} = 0 ,$$

forçant le mouvement à se dérouler dans une hypersurface de l'espace homogène de phase. La condition (3) se ramène pratiquement toujours à la forme

$$(6) \quad p_{f+1} + H = 0 ,$$

où H est l'hamiltonienne ordinaire non homogène éventuellement fonction explicite du temps. Aussi n'est-ce pas H qui est soumis à une conservation, car on a

$$(7) \quad \frac{d}{dt} p_{f+1} = -\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial H}{\partial t} \neq 0 .$$

Mais, c'est là une description particulière au moyen d'un temps singularisé. Une transformation de Lorentz, qui est une transformation canonique, ferait passer à

$$(5') \quad \mathcal{H}' = 0$$

$$(6') \quad p'_{f+1} + H' = 0$$

et à

$$(7') \quad \frac{d}{dt'} p'_{j+1} = - \frac{d\mathcal{H}'}{dt'} = - \frac{\partial H'}{\partial t'}$$

et on ne se trouvera en général pas dans un référentiel où $\partial H'/\partial t' = 0 = dH'/dt'$ ⁽¹⁾.

Alors, (3) n'est pas correctement écrite; ce n'est pas l'énergie E qu'elle doit utiliser, mais p_4 , et l'on doit écrire, conformément à (5),

$$(3^*) \quad \mathcal{H} = p_4^2 - p^2 c^2 - m_0^2 c^4 = 0.$$

On pourrait se demander si le spin est conforme à notre idée d'une grandeur conservée. Il faudrait préciser alors ce qu'on entend par spin. Si c'est un multiple entier ou demi-entier de $\hbar$, c'est un invariant de Galilée et de Lorentz comme la fonction d'action, 2π et les demi-entiers également. Si c'est le vecteur formé à partir des matrices de Pauli ou celles de Dirac, c'est un tenseur-opérateur antisymétrique. Aussi sa conservation est-elle quelque chose d'autre que celle de la charge, bien qu'on puisse envisager une sorte de proportionnalité entre le spin et la charge par l'intermédiaire du moment magnétique lorsque la charge existe. On sait que Blackett avait postulé une proportionnalité universelle entre le moment magnétique et le spin des corps, mais cette hypothèse s'est révélée insoutenable par l'expérience, pour ne rien dire des particules qui se meuvent à la vitesse de la lumière.

Les considérations ci-dessus n'ont peut-être qu'un intérêt pseudo-historique. Mais pour le moins, elles jettent un doute sur le sens qu'il y a à parler strictement de conservation à propos d'autre chose qu'une propriété scalaire. Cela n'est pas dû seulement à ce qu'un scalaire au sens galiléen ne l'est pas nécessairement au sens de Lorentz; mais il est clair de nos jours qu'il faut envisager un invariant de Lorentz (qui sera a fortiori invariant galiléen), susceptible de rester un invariant dans une métrique riemannienne ou même dans le cadre d'une description plus générale, telle une géométrie affine ou un espace de Finsler.

3. — La question se pose alors de savoir exactement ce qu'on entend par conservation. L'idée de conservation contient plus d'anthropomorphisme qu'on ne se le figure en général, surtout en rapport avec l'énergie, car c'est l'énergie qui se paie en argent comptant, opération qui est pour

(1) Le détail de ces calculs se trouve dans A. Mercier: *Principes de Mécanique Analytique* (Paris, 1955).

une bonne part à l'origine psychologique de l'idée d'une équivalence dans les échanges pareils. La conservation physique stricte n'a véritablement de sens que pour une grandeur rigoureusement identique à elle-même et non pour une grandeur qui se scinde en plusieurs catégories.

L'énergie se scinde en plusieurs catégories, et chaque fois qu'un désaccord a été trouvé avec la « conservation », on a imaginé une nouvelle forme d'énergie définie de manière à rétablir la compensation entre toutes les catégories.

La charge, elle, ne se scinde pas en plusieurs formes de la charge. Non seulement elle est scalaire, mais elle est aussi unique.

En outre, l'idée d'une conservation stricte est indissoluble de celle de système isolé. L'isolation est définie même par l'impossibilité de livrer la grandeur à l'extérieur ou de se la fournir de l'extérieur. Il y aura donc autant de types d'isolation qu'il y a de grandeurs strictement conservées.

Il est clair alors que le seul système rigoureusement isolé comporte l'univers dans sa totalité, car le « vide » est sûrement le milieu qui oppose le moins de résistance au transport de quoi que ce soit; c'en serait même la meilleure définition, soit encore, si l'on veut, celle du milieu où les lignes d'univers ont un véritable sens physique, et ces lignes n'ont en principe pas d'extrémités, tout au plus des points d'embranchement, par exemple lors de la création ou l'annihilation de paires de particules. Il n'y a pas d'isolation parfaite à la surface des systèmes soi-disant clos, mais seulement un champ intense simulant une paroi.

Ecrivons alors le théorème de Poynting-Maxwell relatif au seul champ électromagnétique (densité de courant $s^k = 0$)

$$(8) \quad \frac{\partial T_k^i}{\partial x^k} = 0.$$

On sait que $T^{ik} = T^{ki}$, en particulier $T^{i4} = T^{4i}$. Si par projection le long de l'axe du temps, on est conduit à interpréter (T^{14}, T^{24}, T^{34}) comme le vecteur de la densité de quantité de mouvement du champ et (T^{41}, T^{42}, T^{43}) comme celui du flux d'énergie, on les trouve égaux, et l'on peut y entrevoir l'équivalence entre la masse et l'énergie. Si le cas $\nu = 4$ s'interprète comme conservation d'énergie (ou de masse équivalente), cette énergie est d'une nature totalement différente de celles dont il est question en mécanique. Une théorie pareille d'un champ continu non quantifié se suffit à elle-même et il n'y a vraiment aucune raison de lui adjoindre, plutôt qu'une autre force, celle de Lorentz elle-même égalée au taux de variation de la quantité de mouvement de la matière chargée en vertu d'une dynamique faite pour les masses inertes.

En revanche, si l'on établit le même théorème de Poynting-Maxwell en partant des équations de Maxwell avec $s^k \neq 0$, la force de Lorentz

apparaît automatiquement au second membre

$$(9) \quad \frac{\partial T_k^i}{\partial x^i} = s^i F_{ki}$$

comme la seule compatible avec l'interprétation ci-dessus, si l'on appelle quadri-force-densité la grandeur égale à la grande divergence du tenseur de densité énergie-momentum.

Or ces relations concernent des densités. Elles sont donc, déjà dans l'espace pseudo-euclidien de la relativité restreinte, de caractère local. La « force de Lorentz » est donc une conséquence des équations du champ et de l'interprétation énergétique, ou, si l'on veut, elle est le vecteur d'inhomogénéité du théorème quadri-dimensionnel de Poynting.

Mais en écrivant ces choses, — en soi banales et que n'importe quel théoricien a pu établir, — il ressort avec quelque insistance que ni (9), ni (8) ne sont des équations de « conservation » mais qu'elles sont des équations « d'échange » entre des partenaires de natures physiques assez différentes, ici entre le champ et les charges, et, dans une traduction quantique des champs, entre des particules et d'autres de types différents, par exemple des bosons et des fermions.

4. — On peut se demander s'il est naturel et nécessaire de dire que les charges électriques (ϱ et $\mathbf{s}$) sont les « sources » du champ. Les remarques suivantes vont montrer qu'on peut se débarrasser de cette image classique et se rapprocher à la fois du point de vue de la relativité générale et de celui de la théorie quantique des champs.

Lorsqu'on élabore une théorie qui doit dépasser une description déjà interprétée et comprise de phénomènes naturels à l'aide de grandeurs physiques $\{\mathcal{D}\}$ données, il est nécessaire d'ajouter à $\{\mathcal{D}\}$ au moins deux nouvelles grandeurs $\mathcal{G}_1 \mathcal{G}_2$ auxquelles elles sont liées par des relations mathématiques, — deux et non pas seulement une, car s'il n'y en avait qu'une, on aurait une relation $\mathcal{F}(\{\mathcal{D}\}, \mathcal{G}_1) = 0$ et $\mathcal{G}_1$ serait explicitement définie à partir de $\{\mathcal{D}\}$ et on n'aurait jamais de nouvelle théorie capable de décrire plus que jusqu'ici. Si l'on a par exemple déjà interprété les coordonnées d'espace et le temps (si l'on a une cinématique comprenant par exemple les transformations de Galilée ou de Lorentz), on peut introduire à titre d'exemple deux grandeurs que l'on appellera « charge » ($\varrho, \mathbf{s}$) et « champ » ($\mathbf{E}, \mathbf{B}$) et poser $\varepsilon_0 \nabla \cdot \mathbf{E} = \varrho$ etc., soit dans cet exemple les équations de Maxwell. On dit d'habitude que ($\varrho, \mathbf{s}$) est source de ($\mathbf{E}, \mathbf{B}$). Or, de ces équations on déduit mathématiquement (1.b), dite conservation de la charge. Mais en réalité on ne peut pas se donner ($\varrho, \mathbf{s}$), car les fonctions ($\varrho, \mathbf{s}$) résultent d'une équation dynamique utilisant la force de Lorentz, et la force de Lorentz est la seule compatible avec

l'interprétation usuelle (9). On ne peut donc dire qu'une chose, c'est qu'il y a des régions (d'espace-temps) où la grande divergence du champ $(\mathbf{E}, \mathbf{B})$ est nulle et d'autres où elle ne l'est pas :

$$(10.a \text{ ou } b) \quad \frac{\partial F^{ik}}{\partial x^i} \dots \text{Div } F = 0 \quad \text{ou} \quad \varepsilon_0 \text{Div } F = S.$$

En d'autres termes, il y a un cas extérieur et un cas intérieur.

La nécessité d'utiliser ε_0 dans (10.b) ressort de ce que, sans elle, la structure tensorielle de (10.b) serait faussée (à moins de faire artificiellement, de F , une densité) et en outre S n'aurait pas une dimension indépendante mais réductible à celles de la mécanique. T_k^i contient ε_0 lorsqu'on l'exprime au moyen de F .

A côté de (10), on a

$$(11) \quad \sum_{\substack{\text{permutation} \\ ijk}} \frac{\partial F^{ij}}{\partial x^k} = \max F = 0.$$

Cette équation pourrait être attaquée du point de vue ci-dessus selon lequel on ne doit pas relier, une seule nouvelle grandeur, ici F , à des grandeurs déjà interprétées, ici x^k . C'est un fait. Aussi ne peut-on considérer (11) que comme identité devant être vérifiée en plus de (10). C'est une indication que le champ F^{ij} et le champ g^{lk} pourraient n'en faire qu'un dans une théorie unitaire !

Au lieu de (10), on peut utiliser un potentiel φ et poser

$$(12.a \text{ ou } b) \quad \varepsilon_0 \square \varphi = \begin{matrix} 0 \\ \diagdown \\ S \end{matrix}.$$

Comme en relativité générale il est difficile de dire que la matière est la cause de la forme géométrique de la variété riemannienne, il est difficile de dire que la charge est la cause du potentiel. Mais si d'une part $\text{div Div} \equiv 0$, ce qui entraîne la conservation de la charge, $\text{div } \square \neq 0$, si bien que si on écrit (12) pour remplacer (10) et (11), il faut y adjoindre la condition de Lorentz

$$(13) \quad \text{div } \varphi = 0.$$

Autrfois, l'idée d'un potentiel était dérivée de celle de la capacité d'effectuer un travail : l'énergie potentielle. Dans cet ordre d'idée, alors, $-i\varphi^4$ est la partie du quadri-potentiel qui fournira effectivement un travail et $A = (\varphi^1, \varphi^2, \varphi^3)$ celle qui n'en fournira jamais ($\mathbf{s} \cdot \mathbf{s}/c \times \mathbf{B} = 0$). Or, comme il est dit ci-dessus, l'énergie potentielle est une catégorie de signification douteuse.

φ n'est plus interprétable de cette façon. φ est un champ, qui apparaît en certaines régions dites intérieures couplé à un autre champ, celui de la charge; en d'autres régions, il apparaît libre. Le couplage est tel que pour le « champ-charge », la conservation est assurée soit par (10.b) soit par (13). Il est vrai que si d'une part $\operatorname{div} \varphi = 0$ entraîne $\operatorname{div} S = 0$, $\operatorname{div} S = 0$ n'entraîne que $\square \operatorname{div} \varphi = 0$. Aussi la conservation du champ-charge est-elle accompagnée d'une *jauge* du champ-potentiel.

On compare quelquefois la situation valable en relativité générale à celle de l'électrodynamique valable en relativité restreinte. En relativité générale, on a

$$(14) \quad (8\pi\kappa)^{-1}(R_{ik} - \tfrac{1}{2}g_{ik}R) = -T_{ik}.$$

$(8\pi\kappa)^{-1}$ y apparaît comme l'analogue de ϵ_0 , $(R_{ik} - \tfrac{1}{2}g_{ik}R)$ comme celui de $\operatorname{div} F$ et T_k comme l'analogue de S . On a un problème extérieur et un problème intérieur. Mais la comparaison est boiteuse, dans nos connaissances actuelles, car elles se réfère d'abord à des domaines hétérogènes au point de vue dimensionnel comme au point de vue de la structure tensorielle, ensuite à des descriptions géométriques différentes (riemannienne ou euclidienne), et enfin à des types d'interaction peut-être non-unifiables (en dépit de la remarque ci-dessus). En outre, l'électrodynamique classique a fait place à une électrodynamique quantique où les quanta du champ φ sont des photons. Il est moins aisé de s'en référer aux quanta du champ de gravitation.

5. — Dans ces conditions; ne serait-il pas désirable de chercher mieux que de simples analogies, à trouver un terrain commun à la théorie de la gravitation, à celle de l'électromagnétisme et à celles des particules. Jusqu'ici, deux tendances se font jour: réaliser une covariance générale des règles de quantification et quantifier le champ gravifique. Ces deux tendances sont différentes dans leur essence. Peut-être aucune des deux n'est-elle la clef du progrès recherché. Le nombre croissant des particules observées ou postulées et la complexité de leur classification suggèrent que les divers formalismes des théories physiques sont peut-être moins universels qu'on ne le pense par exemple à propos de celui de la relativité générale. On croit qu'il y a quatre types distincts d'interaction parmi lesquels la gravitation seule résiste encore sérieusement à une quantification. Si les gravitons, en tant que particules entremetteuses de l'interaction de gravitation et en tant que quanta du champ gravifique, ont vraiment un sens et une existence autonome, il n'est pas aisé de donner pour le moment, leur propriétés, pour ne rien dire d'une équation qui en régirait le comportement. Tout au plus peut-on leur attribuer une masse nulle (au repos) et un spin. On ne sait pas même s'ils ont, en mouvement, une masse différente de zéro.

Il y a, entre le microscopique et le macroscopique, non seulement une différence d'ordre de grandeur de l'extension des objets, mais aussi et plus encore un renversement dans l'approche de la réalité. Dans le macroscopique, on observe les trajectoires et détermine les interactions qui en rendent compte. Dans le microscopique, on n'observe jamais de véritables trajectoires, et c'est une illusion de croire qu'elles sont rendues visibles dans les chambres à ionisation ou les émulsions photographiques; on ne fait que constater des microcatastrophes que l'on interprète à l'aide de modèles macroscopiques tels que collisions, freinages, etc. A la suite de microcatastrophes pareilles, l'après et l'avant ne sont jamais les mêmes. Nous utilisons des notions issues de notre expérience humaine pour interpréter ces cas qui échappent à nos sens. On construit des modèles sur la constatation d'interactions violentes, et on en cherche les acteurs que l'on nomme particules.

A ce point, on voit mieux qu'autrefois que notre conception de la nature est encore entachée de représentations à l'échelle humaine, et en un sens anthropomorphiques, ce qui ne veut pas dire que l'homme y joue un rôle central, mais l'usage des modèles copiés sur l'observation macroscopique journalière y est encore grand. Le concept de particule, c'est encore celui d'une petite boule à laquelle on attribue des propriétés, un diamètre, une masse et d'autres.

Or, plus on essaie de procéder ainsi, moins le modèle se plie à ce qu'on attend de lui. L'extension spatiale est la plus intuitive, mais peut-être la plus mauvaise de ces déterminations, elle est inaccessible à l'expérience car chaque fois qu'on essaie de l'approcher, la particule nous explose pour ainsi dire dans les doigts. En outre, à cause de la dualité de la matière, ces particules sont non seulement infinitésimales mais elles s'étendent en même temps en tant qu'onde jusqu'aux confins de l'espace. De sorte qu'une attribution spatiale autonome et locale n'a aucun sens. Pour obvier à cette difficulté, il faut avoir recours à des représentations non locales, où certaines propriétés des particules ne sont plus des fonctions ordinaires du point courant de l'espace-temps, mais des fonctionnelles; de façon que l'espace-temps tout entier soit englobé dans la définition de ce qu'on appelle particule, et celle-ci y perd tout caractère rigoureusement local.

C'est pourquoi il se peut que l'essai de localiser des grandeurs dans un entourage microscopique (« presque » infinitésimal) soit voué à un échec. Et si l'une de ces grandeurs était (en un sens non seulement macroscopique mais classique, c'est-à-dire préquantique et même prérelativiste) « conservée », on ne doit pas s'étonner nécessairement qu'il n'en puisse rien être dans une théorie généralement covariante, puisque déjà le presque infinitésimal s'y oppose peut-être pour la raison que l'espace y perd son sens.

D'ailleurs, c'est dès l'apparition de la théorie cinétique puis de la statistique physique que l'on devait s'étonner d'un paradoxe, celui qui résulte de ce que d'une part on manipule des domaines qui, bien que « très » petits, ne sont physiquement pas infiniment petits, de l'autre on assimile ces mêmes domaines à des infinitésimaux au sens du calcul différentiel, et cela avec succès, l'expérience vérifiant la théorie. Cela pouvait aller aussi longtemps que dans un nombre suffisamment grand de cellules se trouvaient groupées des particules en nombre suffisant pour y définir quelque chose comme une densité par une « presque-limite ». Mais lorsqu'on manipule les domaines dont l'extension spatiale est comparable aux sections efficaces des particules, il n'y a peut-être plus de place pour des grands nombres au sens statistique et ce n'est pas en remplaçant les densités par des fonctions delta qu'on se tirera d'affaire. Peut-être n'y a-t-il, en d'autres termes, d'espace physique que pour autant qu'il y a une statistique physique sous-jacente. Il s'en suivrait que l'existence d'une statistique serait la condition de l'emploi possible non seulement de coordonnées, et par extension de coordonnées d'espace-temps, mais de l'emploi des grandeurs canoniquement conjuguées et par conséquent de tout l'appareil dynamique lié à l'introduction de quelque chose du type d'une lagrangienne et à l'existence d'un principe variationnel.

Cela voudrait dire en même temps que la thermodynamique est la condition d'une mécanique, ce qui n'est peut-être pas étonnant car la réversibilité, caractère primitif de toute mécanique, est difficilement concevable autrement que par contraste avec l'irréversibilité.

Mais ce sont là de pures spéculations.

Au cours de transmutations de particules en d'autres, certains principes (non pas des théorèmes) de conservation de type différent de celle de l'énergie-momentum, sont mis en jeu. Ils valent en effet pour des grandeurs quantifiées pour lesquelles ils sont aussi le mieux vérifiées parce que ces conservations sont exactes, se faisant par échange de quanta par nombres entiers ou demi-entiers, de sorte qu'il n'y est pas question d'erreur expérimentale ou d'incertitude. En revanche, les échanges d'énergie-momentum ne se font pas par quanta mais sur le modèle d'une équation différentielle d'espace-temps et sont sujets à l'erreur expérimentale. Cela semble lié à la masse et à son mouvement.

La masse est une grandeur mystérieuse. On ne comprend pas bien comment elle se justifie. En particulier, pourquoi y a-t-il ce spectre compliqué de masses? C'est peut-être un reste encore d'anthropomorphisme qui nous fait classer les particules en première ligne selon leur masse, le graviton, le photon, les leptons, les mésons, les baryons, pour ne rien dire des résonances qui ne sont pas si clairement des « particules » (à moins que les particules ne soient plus ou moins toutes des résonances). Dans cette classification par masse croissante, c'est évidemment un critère

énergétique qui joue, auquel se superpose l'idée de l'état de repos ainsi que l'arrière-pensée d'une grande stabilité reposant sur une cohésion dont l'origine est laissée totalement cachée (sans quoi il devrait y avoir des subparticules).

Certes les groupements de particules se font selon d'autres critères à l'aide de nombres quantiques plus ou moins classiques ou étranges. Mais alors ces classifications plus élaborées ignorent pratiquement tout ce qui tient à la géométrie différentielle traditionnelle. L'axiomatique d'un espace quasi-hilbertien avec quelques groupes appropriés semble suffire pour débiter, donc des espaces abstraits qui nous éloignent beaucoup d'un espace physique au sens traditionnel. La référence à un espace pareil, tout en passant par l'intermédiaire lorentzien, nous rapproche alors beaucoup d'une conception galiléenne, non pas au sens d'un renoncement aux transformations de Lorentz, mais à celui d'une séparation du temps, comme on le voit jusque dans le théorème CPT, séparation imposée moins par les circonstances que par la nature même de la recherche physique — pour ne rien dire de l'irréversibilité foncière du temps qui finit toujours par ressortir par quelque bout.

C'est à se demander si l'idée de conservation, notamment de l'énergie n'est pas étroitement liée à la singularisation du temps qu'on ne peut pas bannir complètement du formalisme canonique, même homogène. Lorsque le temps n'est formellement plus singulier comme il arrive dans une variété quadri-dimensionnelle prise au sérieux, il reste le cône de lumière comme seul signe d'un temps sous-jacent. Or le cône de lumière est lié à ce qu'on appelle la lumière ou du moins à tout ce qui se meut à la vitesse c et qui n'a pas de masse propre tout en étant physiquement existentiel. Nous ne savons pas si c'est la vitesse c qui implique la masse propre nulle ou la masse nulle qui implique la vitesse c ; la séparation de la vitesse et de la masse est éventuellement une maladresse. Il y a là peut-être l'origine d'un problème qu'il est difficile d'énoncer. Le cône de lumière est-il physiquement aussi bien défini que mathématiquement? On n'a jamais que quelque chose comme une matrice S . Et on ne sait pas si les lignes d'univers sont physiquement de vraies lignes ou plutôt des nuages tubulaires.

6. — Il est un problème encore que nous aimerions rappeler parce qu'il fait ressortir une certaine analogie avec ce qui s'est passé avec l'énergie au cours du développement de la physique. C'est celui de l'antimatière qui semble lié à celui de la parité.

La parité étant la propriété de l'état ψ soit de conserver, soit de perdre son signe lors des symétries simples dans l'espace tridimensionnel galiléen, on croyait autrefois que les lois de la physique sont les mêmes, qu'on

les repère dans l'espace ou dans le miroir. L'attribution d'une parité était faite pour satisfaire ce principe.

Voici alors l'analogie. Lorsque, par suite d'un couplage, quelques particules interagissent et se désintègrent en d'autres, la parité après la désintégration est soit identique, soit contraire à ce qu'elle était avant. Si elle est la même, elle est conservée. Sa conservation a été trouvée en défaut dans l'émission des neutrinos par les particules leptoniques. D'autres principes de physique ont subi un sort analogue. Plusieurs fois, celui de conservation de l'énergie a risqué de s'effondrer mais a été sauvé par la découverte d'une forme nouvelle d'énergie.

On estime pouvoir sauver la situation en disant que la condition de conservation par rapport à une symétrie simple est trop restrictive tout comme il est trop restrictif de ne considérer que de l'énergie mécanique par exemple, et qu'il y a une conservation valable pour le couplage leptique par rapport à une « symétrie » élargie qui consiste, simultanément, à échanger la gauche et la droite et à remplacer la matière par son antimatière. On vérifie par le calcul l'invariance correspondante des lois de la désintégration leptique, mais on ne comprend pas bien pourquoi. On le saura peut-être, le jour où le vrai rapport de la matière à l'anti-matière nous sera devenu familier. Quoi qu'il en soit, il faudra vraisemblablement inclure ce rapport comme d'autres jusqu'ici relatifs aux transformations de Lorentz seulement, à une description généralement covariante ou mieux encore dans la superthéorie qui unifiera le champ de gravitation aux autres champs des interactions connues, ce qui n'ira pas sans de considérables modifications de nos vues fondamentales.

Cette référence au problème de la conservation de la parité n'a été faite que pour nous rendre prudents et confiants à la fois. Il faut toujours s'attendre à ce que des difficultés surgissent même dans la poursuite d'une idée aussi bien ancrée que celle de conservation. Mais en même temps, il ne faut jamais perdre l'espoir de les vaincre par une réinterprétation audacieuse.

7. — Les remarques qui précèdent, critiques et disparates, ne laissent pas grand' chose de positif. Il est malaisé de conclure sur une note constructive.

On aurait pu se demander si Galilée, à la mémoire duquel sont dédiés nos débats, a contribué à poser ou à préciser le problème de la conservation. Apparemment non. Néanmoins, on lui doit un pas important, celui d'avoir isolé le « cas pur » par l'élimination d'influences perturbatrices plus ou moins étrangères, en particulier les frottements. D'où la « loi » d'inertie, cas le plus simple de conservation, mais de conservation de la quantité de mouvement. C'est donc lui qui a éclairci le point de départ de toute considération sur la conservation. Mais les influences

perturbatrices ne s'éliminent que par la restriction à un cas pur, elles existent bel et bien, et il faut les faire rentrer dans un cadre élargi qu'on a appelé la dynamique. Et le problème s'est posé d'étudier la conservation ou la non conservation dans des dynamiques toujours plus élargies. Y a-t-il une Dynamique qui englobe toutes les autres? Ce serait théorie unitaire! La succession qui a mené de la mécanique rationnelle à celle des corps déformables, à l'électrodynamique, à celle de la relativité, à celle des quanta et à la mécanique statistique semble laisser la place à deux possibilités: soit une suite de dynamiques plus ou moins apparentées et se généralisant ou se complétant les unes les autres sans qu'il y ait cependant une Dynamique capitale vers laquelle tende la science, comme si, plus ou moins, chaque interaction nécessitait sa description particulière, d'où une physique hétérogène, — soit justement, une tendance unitaire au bout de laquelle on aura cette Dynamique unique.

Mais il ne faut pas se faire d'illusion. Une Dynamique finale serait du même coup un arrêt final et l'opposé d'une dynamique au sens ci-dessus. On est sûrement encore loin d'y parvenir, quand on pense aux problèmes que posent les interactions forte, faible, et d'autres qui règlent les échanges entre les multiples particules. Si bien que la première éventualité semble, à l'heure actuelle, plus proche de la réalité que la seconde.

Du point de vue de la discussion qui précède, on pourrait énoncer, pour thèse, que la tâche consiste à élaborer une théorie, en un sens unitaire, dans laquelle toutes les « conservations » seraient des théorèmes conséquences des postulats fondamentaux. En mécanique newtonienne des systèmes non dissipatifs, c'est le cas; la conservation de l'énergie, celle de la quantité de mouvement et celle du moment de rotation sont cependant distinctes. En relativité restreinte, c'est aussi le cas, mais les deux premières n'en font qu'une, tensorielle.

Il y aurait comme deux points de vue de l'unification des théories. (a) L'unification des théorèmes généraux: chaque théorie, basée sur des postulats disparates, admettrait cependant les mêmes théorèmes. Par exemple, la théorie newtonienne de la gravitation et la théorie maxwellienne de l'électromagnétisme admettraient la conservation de l'énergie, compte tenu de la radiation dans la seconde. (b) L'unification au sens ordinaire en une théorie unitaire d'un superchamp groupant, en une seule loi, des lois d'interaction jusqu'ici disparates.

Mais comme nous venons de le suggérer, il faut remarquer que l'unification (b) tend vers l'élimination finale de toute base dynamique. En effet, une dynamique est une théorie dans laquelle on a quelque chose comme une force d'accélération, une lagrangienne dans un principe de variation ou une hamiltonienne responsable des trajectoires de phase. Force, lagrangienne ou hamiltonienne y figurent comme grandeurs théoriques non spécifiées. Leur spécification par telle ou telle « loi de force »

revient à l'étude de telle ou telle interaction: gravitationnelle, électromagnétique, faible, forte Il y a des interactions fondamentales comme dans ces quatre exemples. Il y a aussi des modèles, comme l'oscillateur ou toute loi que l'on peut se donner arbitrairement. Parmi elles, l'approximation linéaire en fonction des paramètres choisis joue un rôle fréquent. Autrement dit, une dynamique de ce genre est construite pour y faire jouer diverses interactions qui n'ont de commun que le fait de rentrer dans le même schème de description, mais qui ne sont pas unifiées pour autant.

Une théorie unifiée, même issue d'une théorie de type dynamique avec, par exemple, un principe de variation, ne devrait plus contenir rien qui fût arbitraire, c'est-à-dire être telle que la lagrangienne ne soit plus laissée libre à aucun choix, ce qui revient à esquiver un principe de variation ou à le rendre équivalent à la « seule » loi d'interaction possible. C'est un peu ce que fait la relativité générale, pour autant qu'elle se confine à l'étude de la gravitation. Parallèlement, la conservation d'énergie-momentum n'est plus alors un théorème conséquent au principe de variation ni une intégrale première, mais l'interprétation physique d'identités vérifiées par certaines grandeurs convenablement définies. Dans ces identités, il y a alors quelque chose de satisfaisant mais aussi d'arbitraire: satisfaisant dans la nature d'identité, arbitraire dans la gratuité de la définition des dites grandeurs.

Dans une théorie unifiée au sens (b), il ne serait plus désirable qu'on ait à choisir une loi caractéristique d'interaction. Il faudrait donc que ce soit un modèle univoque où tout serait nécessaire, un modèle véritablement universel, auquel toute idée dynamique serait devenue étrangère. Du même coup, il n'y figurerait plus aucune nature spécifiquement physique, sauf dans la nécessité d'accepter ce modèle universel. Ce serait la réalisation d'un superdéterminisme, dans lequel aucune cause efficiente, à savoir aucune cause initiale, n'aurait de place. Il n'y aurait plus qu'une cause formelle. Alors on devrait pouvoir y faire deux choses: déterminer d'une part une ou plusieurs constantes universelles, nombres qui seraient comme les mesures fixes et nécessaires de l'univers, telle peut-être la constante de la structure fine; construire d'autre part une ou plusieurs grandeurs qui satisferaient des identités, telle la conservation de la charge. Ces identités auraient éventuellement diverses formes. Telle ou telle d'entre elles s'interpréterait comme « conservation » de quelque chose. Mais des identités pareilles n'ont plus d'intérêt synthétique ou constructif. Elles ne seraient que des jugements analytiques. Peut-être auraient-elles de l'intérêt pour l'expérience ou la pratique. Car cette dernière aurait alors deux buts à poursuivre: déterminer les constantes universelles et vérifier les identités. La détermination de constantes universelles est une tâche traditionnelle de la physique expérimentale; elle en est même la

seule irréductible à la théorie. La vérification des identités aurait ceci de satisfaisant qu'on aurait affaire à une interprétation de la théorie dans la pratique, mais elle aurait aussi ceci d'arbitraire que des identités sont des trivialités et que leur interprétation expérimentale ne découvre rien du tout et qu'on ne saurait pas si l'interprétation expérimentale qu'on en a faite n'est pas, elle aussi, une banalité.

8. — Dans l'état où la physique se trouve à la fin du deuxième tiers de ce siècle, il y a en gros trois sortes de relations que l'on affuble du nom de conservation. D'une part, il y a ce que nous avons appelé véritable conservation, celle de la charge. Ensuite, il y a celle du type d'énergie-momentum que nous avons critiquée. Enfin, en théorie des particules, il y a les conservations de nombres superquantiques plus ou moins étranges, baryonombre, parité, etc.

Là encore, il y a cet aspect disparate. Mais si ces relations devaient finalement prendre la forme d'identités, il n'y a aucun inconvénient à cela. Il nous faut attendre pour voir ce que l'avenir nous réserve.

Cette conclusion est vague. Notre but n'était pas d'élaborer une théorie jusque dans ses conséquences détaillées. Pour excuse, rappelons que nous ne voulions autre chose que faire le point sur le plan de la philosophie des sciences, ce qui laisse toujours le théoricien pur insatisfait.

INTERVENTI E DISCUSSIONI

— J. TREDER:

The break-down of the conservation laws for energy-momentum etc. in the general Riemann manifold of general relativity has another meaning (physical and mathematical) as the break-down of the conservation laws for hypercharges and so on in the case of weak interactions. In this case we are able to disprove the conservation laws. But in general relativity we cannot disprove the conservation of energy etc. That has the sophistical cause that in general relativity a meaningful definition of energy-momentum etc., is leased—according to Einstein—on conservations. Therefore we have the sophistical point that in general relativity the quality «energy» does not exist if for this quality no conservation law is existent.

— A. MERCIER:

I agree with Treder upon the difference between the two cases of conservation which he distinguishes and am thankful to him to have made still clearer than I did my own point, *viz.* that energy, momentum and the like can in general relativity be defined only insofar as conservation theorems can be established for these quantities.

— P. G. BERGMANN:

1) We must distinguish between symmetry and conservation. The relationship between the two depends on an action principle.

2) The notion of the differentiable manifold is undoubtedly anthropomorphic. Whether it should be modified remains to be seen.

3) The ultimate break-down of symmetry may be conditioned by cosmological considerations.

— A. MERCIER:

(*Remark 2*)

Yes, this remains to be seen, but I did not advocate that one should drop altogether the notion of a differentiable manifold, I simply doubted that the concept of a physical space in the Kantian tradition still has a meaning for so-called sub-atomic dimensions. However conversely, in this connexion, I might put a question to either Bergmann or an experienced mathematician like Lichnerowicz: is it in your opinion necessary to keep a mathematical notion of a manifold (in contradistinction to that of a physical space) as a background reference in order to build up a physical theory, or are these indications that something like plain topology will suffice?

(*Remark 3*)

I agree with the suggestion that a break-down of symmetry may come from cosmology; there are perhaps indices already now, but I am not enough of a cosmologist to answer that.

— N. ROSEN:

In connection with the question of the existence of a variational principle, I do not know whether this implies something fundamental or whether it is just a matter of mathematical ingenuity to find a variational principle for given equations of motion. If one assumes that there exists such a principle, it is true that one can ask why this particular one and not some other one. However, one can equally well ask why *these* equations of motion and not some others. The following thought has occurred to me (and I am sure that others have thought of it).

One can conceive of many different universes existing side by side, each based on different variational principles or equations of motion. What distinguished our universe from all others is that in this case the equations of motion to the possibility of the existence of conscious beings who are aware of the universe, which in all other cases no such possibility exists.

— A. MERCIER:

Prof. Rosen's argument is most interesting, especially in its third part, and I am grateful for the suggestion made which might be appreciated as a modern expression of Leibniz's argument about «the best of all possible worlds».

— P. G. BERGMANN:

Though formally action principles can be constructed, the conclusion (Noether's theorems) may be trivial.

— G. E. TAUBER:

Prof. Mercier's ideas are certainly very exciting and stimulating. I am just wondering how serious the break-down of parity should be taken, since there is absolutely no evidence at present that parity is not conserved in strong interaction. I would also like to ask how we can formulate irreversible thermodynamics in a covariant formalism, or whether we have to give up the concept of covariance, as Bergmann seems to indicate.

— A. MERCIER:

As regards the first question of Dr. Tauber, I may recall that I indicated the way followed by physicists who try to save parity conservation even for weak interactions.

As to the formulation of thermodynamics, I can only refer to Stueckelberg's attempt and emit the hope that sometime people will be able to elaborate a thermodynamics worthy of that name, *i.e.* valid not only for phenomena artificially made reversible.

— P. HAVAS:

Maybe the importance of Noether's theorem for the problem of the existence of conservation laws has been overrated. From this theorem one obtains only differential laws of the form (1). These correspond to true conservation laws of the form (1') only if the flux at infinity vanishes, which is not necessarily the case for all such differential laws. Also, while Noether's theorem provides a very convenient tool to obtain such differential laws if an action principle is available, there exist methods (due to Lie and Engel) to obtain them directly from the invariance properties of the differential equations of a theory, whether or not these equations are derivable from an action principle.

— A. MERCIER:

Dr. Havas' intervention seems to be an answer to Bergmann's last remark rather than an objection to my arguments.

— M. A. TONNELAT:

1) L'interaction champ-particule qu'exprime dans (9) la force de Lorentz $s^i F_{ki}$ est impliquée par la notion même d'énergie du champ total, grandeur quadratique, qui comprend outre la somme des énergies de chaque champ (champ extérieur, champ propre de la particule) un terme énergétique d'interaction. C'est bien la divergence d'un terme d'interaction entre champs qui correspond à l'ancienne notion de force.

Techniquement

$$\partial_k [(T^k_i)_{\text{particule}} + t^k_j ((F + F')^2) - t^k_j (F'^2)] = 0$$

F , champ extérieur; F' , champ propre,

est équivalent à

$$\partial_k [T^k_i + t^k_j (F^2) + t^k_j (F \cdot F')] = 0.$$

C'est pourquoi même si les équations du champ sont linéaires, une loi de conservation appliquée à l'énergie globale, implique nécessairement une interaction sous-jacente.

2) L'unification en un superchamp tend, selon le Professeur Mercier, à l'élimination finale de toute dynamique. La théorie du champ unifié d'Einstein-Schrödinger n'en fournit-elle pas le meilleur exemple? La négation d'un dualisme champ-matière c'est à dire l'absorption de la matière par un superchamp conduit à la difficulté de savoir ce qui se meut et d'en prévoir le mouvement (terme d'accélération).

Ce serait une illustration technique de l'affirmation plus philosophique énoncée ici.

— A. MERCIER:

La première remarque de M.me Tonnelat est une contribution à mon article en généralisant un argument.

La seconde est une question difficile. Je crois pouvoir répondre par une affirmative provisoire, tout en faisant une réserve. Il semble bien que l'effort concerté d'Einstein-Schrödinger tend vers un type de théorie unitaire d'où l'élément dynamique serait éliminé. Mais la théorie d'Einstein-Schrödinger n'est certes pas la théorie ultime qui coupe court à tout développement ultérieur, puisqu'elle ne contient rien qui concerne les problèmes des particules et les interactions forte et faible. Aussi la question posée reste-t-elle inévitablement sans réponse définitive.

Local Tetrad in General Relativity.

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1. - Introduction.

I have given elsewhere [1] a general method elaborated by Mr. M. Cahen, Miss L. Defrise and myself, for computing the curvature tensor and equation of gravitational field. It is another version of the spin coefficients method of E. Newman and R. Penrose [2]. I shall give a summary of the method and apply it, mainly, to the problem of determining all homogeneous empty space solutions.

2. - A vector representation of the curvature form.

V_4 denotes a four dimensional Riemannian space subjected to usual differentiability conditions.

The parallelism of Levi-Civita induces a mapping p of the tangent space in a point x onto the tangent space in a neighbouring point $x+dx$.

The tangent space is a Minkowski space, and the mapping p preserves the metric structure of tangent spaces.

If θ^a is a base of 1-forms of the tangent space, or *local tetrad*, let

$$(2.1) \quad ds^2 = g_{ab} \theta^a \theta^b \quad a, b = 0, 1, 2, 3 .$$

The mapping p is thus an infinitesimal Lorentz transformation, more precisely p is determined by a 1-form ω with value in the Lie-algebra of the Lorentz-group.

In the following we restrict ourself to the homogeneous Lorentz transformations of the group $\mathcal{L}_+^\uparrow$ preserving the orientation of the four space, and the orientation of time directions.

The local components of ω are

$$(2.2) \quad \omega^a_b = \gamma^a_{bc} \theta^c .$$

Conservation of the metric structure of tangent space is expressed by

$$(2.3) \quad \omega_{ab} + \omega_{ba} = dg_{ab}.$$

Furthermore, Levi-Civita parallelism is such that (null torsion)

$$(2.4) \quad d\theta + \omega \wedge \theta = 0 \quad \text{or} \quad d\theta^a + \omega^a_b \wedge \theta^b = 0.$$

If R is any representation of $\mathcal{L}_+^\dagger$, let

$$(2.5) \quad \sigma = R\omega.$$

The representation K we make use of is the representation

$$(2.6) \quad K: \mathcal{L}_+^\dagger \rightarrow O_3^+(C)$$

of $\mathcal{L}_+^\dagger$ in the group $O_3^+(C)$ of proper orthogonal transformations of a 3-dimensional complex space, denoted in following by E_3 .

K can be realized as the representation of $\mathcal{L}_+^\dagger$ in the space of real bivectors, or the space of real 2-forms: M_6 .

One can use as a base of M_6 , three complex self dual 2-forms, and their complex conjugates

$$(2.7) \quad Z^\alpha, \bar{Z}^\alpha \quad \alpha = 1, 2, 3$$

where

$$(2.8) \quad Z^{\alpha*} = iZ^\alpha$$

$*$ is the dual operator.

Any real 2-form F , or associated bivector F_{ab} , reads as follows

$$(2.9) \quad F = \frac{1}{2} F_{ab} \theta^a \wedge \theta^b = F_\alpha Z^\alpha + \bar{F}_\alpha \bar{Z}^\alpha.$$

Any transformation of $\mathcal{L}_+^\dagger$ induces a rotation in E_3 . Let

$$(2.10) \quad \gamma_{\alpha\beta} Z^\alpha Z^\beta$$

be the metric quadratic form for E_3 .

In our case $\sigma = K\omega$ is an infinitesimal rotation, *i.e.*, a vector of E_3 . If σ^α_β are the components of σ , let

$$(2.11) \quad \sigma^\gamma = \epsilon^{\gamma\alpha\beta} \gamma_{\alpha\delta} \sigma^\delta_\beta$$

be the component of the *vector* associated with σ .

The curvature 2-form

$$(2.12) \quad \Omega = d\omega + \omega \wedge \omega \leftrightarrow \Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$$

being also an infinitesimal Lorentz transformation, gives also rise to a vector

$$(2.13) \quad \Sigma = K\Omega.$$

Let

$$(2.14) \quad \Sigma^\gamma = \Sigma^{\gamma\alpha\beta} \gamma_{\alpha\delta} \Sigma^\delta_\beta.$$

As any 2-forms, Σ_α can be expressed as follows:

$$(2.15) \quad \Sigma_\alpha = C_{\alpha\beta} Z^\beta + E_{\alpha\bar{\beta}} \bar{Z}^\beta + \frac{\lambda}{3} \gamma_{\alpha\beta} Z^\beta, \quad \gamma^{\alpha\beta} C_{\alpha\beta} = 0.$$

Because of (4) one can check that

$$(2.16) \quad C_{\alpha\beta} = C_{\beta\alpha},$$

and for reality conditions

$$(2.17) \quad E_{\alpha\bar{\beta}} = \bar{E}_{\bar{\beta}\alpha}$$

$C_{\alpha\beta}$ is thus a symmetric quadratic form, related to Weyl conformal curvature tensor, $E_{\alpha\bar{\beta}}$ an hermitian matrix, related to Ricci tensor, λ a scalar related to the scalar curvature $\lambda = -R/2$.

Empty spaces are such that

$$(2.18) \quad E_{\alpha\bar{\beta}} = \lambda = 0$$

or

$$(2.19) \quad \Sigma_\alpha \wedge \bar{Z}_\beta = 0, \quad \Sigma_\alpha \wedge \Sigma^\alpha = 0.$$

This method can be developped for any choice of local tetrad: orthonormal, isotropic, a.s.o.

Let us give some more details about *isotropic tetrads*. One can write

$$(2.20) \quad ds^2 = 2\theta^0\theta^3 - 2\theta^1\theta^2$$

where θ^0, θ^3 are real forms, θ^1, θ^2 complex conjugate forms.

As a base for M_6 we use

$$(2.21) \quad Z^1 = \theta^2 \wedge \theta^3 \quad Z^2 = \theta^0 \wedge \theta^1 \quad Z^3 = \frac{1}{2}(\theta^0 \wedge \theta^3 - \theta^1 \wedge \theta^2).$$

One has

$$(2.22) \quad \gamma_{\alpha\beta} Z^\alpha Z^\beta = 2Z^1 Z^2 - 2(Z^3)^2$$

$$(2.23) \quad dZ + \sigma \wedge Z \leftrightarrow dZ^\alpha + \sigma^\alpha_\beta \wedge Z^\beta = 0$$

$$(2.24) \quad \Sigma = d\sigma + \sigma \wedge \sigma \leftrightarrow \Sigma_{\alpha\beta} = d\sigma_{\alpha\beta} + \sigma_{\alpha\gamma} \wedge \sigma^\gamma_\beta$$

and

$$(2.25) \quad \Sigma_1 = d\sigma^2 - \sigma^2 \wedge \sigma^3, \quad \Sigma_2 = d\sigma^1 + \sigma^1 \wedge \sigma^3, \quad \Sigma_3 = d\sigma^3 + \frac{1}{2}\sigma^1 \wedge \sigma^2.$$

Finally Bianchi Identities take the form

$$(2.26) \quad d\Sigma + \sigma \wedge \Sigma - \Sigma \wedge \sigma = 0$$

or

$$(2.27) \quad \begin{cases} d\Sigma_1 + \Sigma_1 \wedge \sigma^3 + \frac{1}{2}\Sigma_3 \wedge \sigma^2 = 0, \\ d\Sigma_2 - \Sigma_2 \wedge \sigma^3 - \frac{1}{2}\Sigma_3 \wedge \sigma^1 = 0, \\ d\Sigma_3 - \Sigma_1 \wedge \sigma^1 + \Sigma_2 \wedge \sigma^2 = 0. \end{cases}$$

3. - Homogeneous empty spaces.

3'1. Definition. — We define as *homogeneous*, spaces such that any scalar invariant has constant value. This is of course the case for spaces which admit a transitive group of motion. One can search among the known results about groups of motion empty spaces solutions, a review of which can be find in PETROV [3].

There exists only solutions of type *I* and type *N*, Petrov having exhibited only a special case of type *N*.

3'2. Type I. — One can indeed define various « canonic » tetrads.

We use for our purpose a local isotropic tetrad such that

$$(3.1) \quad C_{\alpha\beta} = \begin{pmatrix} A & B & 0 \\ B & -A & 0 \\ 0 & 0 & 4B \end{pmatrix}.$$

In the case of empty spaces, Bianchi identities take the form:

$$(3.2) \quad \begin{cases} A_{,0} + A(2\sigma_0^3 + \frac{1}{2}\sigma_2^1) - \frac{3}{2}B\sigma_2^2 = 0, \\ A_{,1} + A(2\sigma_1^3 + \frac{1}{2}\sigma_3^1) - \frac{3}{2}B\sigma_3^2 = 0, \\ A_{,2} - A(2\sigma_2^3 + \frac{1}{2}\sigma_2^1) - \frac{3}{2}B\sigma_0^1 = 0, \\ A_{,3} - A(2\sigma_3^3 + \frac{1}{2}\sigma_3^1) - \frac{3}{2}B\sigma_1^1 = 0, \\ B_{,0} + \frac{3}{2}B\sigma_2^1 + \frac{1}{2}A\sigma_2^2 = 0, \\ B_{,1} + \frac{3}{2}B\sigma_3^1 + \frac{1}{2}A\sigma_3^2 = 0, \\ B_{,2} - \frac{3}{2}B\sigma_0^2 + \frac{1}{2}A\sigma_0^1 = 0, \\ B_{,3} - \frac{3}{2}B\sigma_1^2 + \frac{1}{2}A\sigma_1^1 = 0. \end{cases}$$

With respect to a canonical tetrad, all elements such as A, B, σ_a^α are *constants*.

We can solve (3.2) in the following way:

$$(3.3) \quad \begin{cases} \sigma^3 = (a, b, c, d), \\ \sigma^1 = (-A^1c, -A^1d, -B^1a, -B^1b) \\ \sigma^2 = (-B^1c, -B^1d, A^1a, A^1b) \end{cases}$$

with

$$(3.4) \quad A^1 = \frac{12AB}{A^2 + 9B^2}, \quad B^1 = \frac{4A^2}{A^2 + 9B^2}$$

$A^2 + 9B^2 = 0$ implies the type D , which is excluded by hypothesis. The four constants a, b, c, d are subjected to empty space equations, which form a system of algebraic quadratic equations.

The discussion of this system depends on the value of B^1 .

If $B^1 \neq 1$ it is possible to show that the space must necessarily be of type D .

If $B^1 = 1$, $A^1 = \pm \sqrt{3}$, and

$$(3.5) \quad a = d = 0 \quad b^2 = c^2.$$

$$(3.6) \quad \text{If } b = -c \quad b \text{ is real} \quad \text{and} \quad b^2 = -\frac{B}{2} \quad (B < 0).$$

$$(3.7) \quad \text{If } b = c \quad b \text{ is purely imaginary } b = i\beta, \beta^2 = -\frac{B}{2};$$

(3.6) and (3.7) correspond to the same solution with opposite signature.

Let $b = -c$, one can write

$$(3.8) \quad \begin{cases} \sigma^3 = \frac{-k}{\sqrt{2}} (\theta^1 - \theta^2), \\ \sigma^2 = -\frac{k}{\sqrt{2}} (\theta^0 + \sqrt{3}\theta^3), \\ \sigma^1 = -\frac{k}{2} (\sqrt{3}\theta^0 - \theta^3), \end{cases}$$

$$k = \sqrt{-B};$$

$$(3.9) \quad \begin{cases} d\theta^1 = -d\theta^2 = \frac{k}{\sqrt{2}} \theta^1 \wedge \theta^2, & d(\theta^1 + \theta^2) = 0, \\ d\theta^0 = \frac{k}{2\sqrt{2}} (\theta^1 + \theta^2) \wedge (\theta^0 + \sqrt{3}\theta^3), \\ d\theta^3 = \frac{k}{2\sqrt{2}} (\theta^1 + \theta^3) \wedge (\theta^3 - \sqrt{3}\theta^0). \end{cases}$$

If

$$(3.10) \quad \theta^1 + \theta^2 = \frac{\sqrt{2}}{2} dx^1$$

one finally has as solution

$$(3.11) \quad ds^2 = \exp[kx^1] [\cos \sqrt{3} kx^1 ((dx^0)^2 - (dx^3)^2) + 2 \sin \sqrt{3} kx^1 dx^0 dx^3] - (dx^1)^2 - \exp[-2kx^1] (dx^2)^2.$$

This space has the following properties.

It is the only empty space solution of type I such that the *four characteristic isotropic directions* form an *equianharmonic range*. The four range has only two distinct values $\frac{1}{2}(1 \pm i\sqrt{3})$.

This correspond to the maximum symmetry of four isotropic directions in four space.

It is the only empty space solution wich admits a G_4 transitive group of motion.

With respect to the above co-ordinates, the Killing vectors are

$$(3.12) \quad \begin{cases} X_1 = (\delta_{(0)}^a), & X_2 = (\delta_{(2)}^a), & X_3 = (\delta_{(3)}^a) \\ X_4 = \left(-\frac{k}{2} (x^0 + \sqrt{3}x^3) \delta_{(0)}^a, & \delta_{(1)}^a, & k\delta_{(2)}^a, & \frac{k}{2} (\sqrt{3}x^0 - x^3) \delta_{(3)}^a \right), \end{cases}$$

with the following commutation relations

$$(3.13) \quad \left\{ \begin{array}{l} [X_\alpha X_\beta] = 0 \quad [X_1 X_4] = -\frac{k}{2}(X_1 - \sqrt{3}X_3), \\ [X_3 X_4] = -\frac{k}{2}(\sqrt{3}X_1 + X_3), \\ [X_2 X_4] = kX_2. \end{array} \right.$$

If

$$(3.14) \quad u = \frac{x^3 + ix^0}{\sqrt{2}}$$

and $\varepsilon_1 = \frac{1}{2}(1 + i\sqrt{3})$, $\varepsilon_3 = \frac{1}{2}(1 - i\sqrt{3})$, $\varepsilon_2 = -1$, $\varepsilon_4 = 0$

$$(3.15) \quad ds^2 = -(\exp[\varepsilon_4 x^1] dx^1)^2 - (\exp[\varepsilon_1 x^1] du)^2 - \\ - (\exp[\varepsilon_3 x^1] d\bar{u})^2 - (\exp[\varepsilon_2 x^1] dx^2)^2.$$

The four complex numbers ε_a also form an equianharmonic range, and with respect to the new variables:

$$(3.16) \quad [X_\alpha X_4] = -\varepsilon_\alpha X_\alpha \quad \alpha = 1, 2, 3, 4.$$

3.3. Type N. — If θ^0 is associated with the four fold characteristic isotropic vector, $C_{\alpha\beta}$ takes the following form

$$(3.17) \quad C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The tetrad is determined up to a Lorentz transformation of a G_4 group, which operates on the θ^a as follows:

$$(3.18) \quad \left\{ \begin{array}{l} \theta^{0'} = \exp[a] \theta^0, \\ \theta^{1'} = \exp[ib](\theta^1 + \bar{\gamma}\theta^0), \\ \theta^{2'} = \exp[-ib](\theta^2 + \gamma\theta^0), \\ \theta^{3'} = \exp[-a](\theta^3 + \gamma\theta^1 + \bar{\gamma}\theta^2 + \gamma\bar{\gamma}\theta^0), \end{array} \right.$$

and on the Z^α in the following way

$$(3.19) \quad \left\{ \begin{array}{l} Z^{1'} = \exp[-w](Z^1 + \gamma^2 Z^2 + 2\gamma Z^3), \\ Z^{2'} = \exp[w] Z^2 \\ Z^{3'} = \gamma Z^2 + Z^3, \end{array} \right. \quad w = a + ib,$$

γ is a complex coefficient, a, b real coefficients.

We also have

$$(3.20) \quad \begin{cases} \sigma'^1 = \exp[-w](\sigma^1 + \gamma^2 \sigma^2 + 2\gamma \sigma^3 - 2d\gamma), \\ \sigma'^2 = \exp[w] \cdot \sigma^2, \\ \sigma'^3 = \sigma^3 + \gamma \sigma^2 - dw, \end{cases}$$

$$(3.21) \quad C'_{22} = \exp[-2w] C_{22}.$$

The G_4 group can be restricted to a sub-group G_2 by the conditions

$$(3.22) \quad C_{22} = 1, \quad w = 0.$$

It follows that σ^2 is an invariant 1-form.

Bianchi identities imply

$$(3.23) \quad \sigma^2 \wedge Z^2 = 0, \quad 2\sigma^3 \wedge Z^2 = \sigma^2 \wedge Z^3$$

or

$$(3.24) \quad \sigma_2^2 = \sigma_3^2 = 0, \quad \sigma_0^2 = -4\sigma_2^3, \quad \sigma_1^2 = -4\sigma_3^3.$$

The first condition ($\sigma^2 \wedge Z^2 = 0$) means that the characteristic congruence associated with θ^0 , in short the congruence θ^0 , is geodesic and shear free.

The transformation of the components of σ^2 under the group G_2 can be written:

$$(3.25) \quad \sigma'^2_1 = \sigma^2_1, \quad \sigma'^2_0 + \bar{\gamma} \sigma'^2_1 = \sigma^2_0$$

σ^2_1 is a scalar invariant. Two cases are to be investigated:

a) $\sigma^2_1 \neq 0$, from homogeneity and the algebraic empty spaces equations it follows $\sigma^2_1 = 0$;

b) $\sigma^2_1 = 0$, σ^2_0 is a scalar invariant, and again from empty space equation it follows $\sigma^2_0 = 0$.

As a consequence, empty homogeneous spaces on type N are such that:

$$(3.26) \quad \sigma^2 = 0.$$

This condition expresses that the vector field θ^0 is a *parallel field*, which gives a well known type of space.

If furthermore these spaces are supposed to be empty, they are *plane wave with parallel rays solutions*.

If $\sigma^2 = 0$

$$(3.27) \quad \Sigma_1 = 0, \quad \Sigma_2 = d\sigma_1 + \sigma^1 \wedge \sigma^2, \quad \Sigma_3 = -2d\sigma^3 = 0.$$

If we go back to (3.20)

$$(3.28) \quad \sigma'^3 = \sigma^3 - dw$$

the last condition (3.27) tells us that one can always choose w such that

$$(3.29) \quad \sigma^3 = 0, \quad dw = 0.$$

We have again from (3.20), (3.26), (3.29)

$$(3.30) \quad \sigma'^1 = (\sigma^1 - 2 d\gamma) \exp[-w].$$

Among the empty space equations

$$(3.31) \quad d\sigma^1 = C_{22} \cdot Z^2$$

are precisely those integrability conditions which permit to write

$$(3.32) \quad \sigma_1^1 = \sigma_2^1 = \sigma_3^1 = 0, \quad \sigma^1 = \sigma_0^1 \theta^0.$$

Finally:

$$(3.33) \quad d\theta^0 = d\theta^1 = d\theta^2 = 0, \quad d\theta^3 = \sigma_0^1 \theta^0 \wedge \theta^1 + \text{conj}.$$

This corresponds to the ds^2

$$(3.34) \quad ds^2 = -dz d\bar{z} + 2 du(dv + H(z, \bar{z}, u) du),$$

with

$$(3.35) \quad H_{z\bar{z}} = 0,$$

$$(3.36) \quad \theta^0 = du, \quad \theta^1 = dz, \quad \theta^3 = dv + H du, \quad \sigma^1 = 2H_z \cdot \theta^0, \quad \sigma^2 = \sigma^3 = 0,$$

z is a complex variable, H is an analytic function.

We have

$$(3.37) \quad C_{22} = -2H_{zz} \cdot \theta^0 \wedge \theta^1.$$

Because of

$$(3.38) \quad C'_{22} = \exp[-2w] C_{22}$$

where w is a constant

$$(3.39) \quad \varrho = d \ln C_{22}$$

is an invariant 1-form, and because of Bianchi identities

$$(3.40) \quad \varrho \wedge Z^2 = 0$$

thus

$$(3.41) \quad \varrho = \varrho_0 \theta^0 + \varrho_1 \theta^1,$$

with transformation laws

$$(3.42) \quad \begin{cases} \exp[a] \varrho'_0 + \exp[ib] \bar{\gamma} \varrho'_1 = \varrho_0, \\ \exp[ib] \cdot \varrho'_1 = \varrho_1. \end{cases}$$

Let $\varrho_1 = 0$, this means

$$(3.43) \quad H_{zzz} = 0 \quad H = F(u) \cdot z^2 + \text{conj}$$

and

$$(3.44) \quad H_1 = \varrho'_0 |C_{22}|^{-\frac{1}{2}}$$

is a scalar invariant.

If

$$(3.45) \quad H_1 = \alpha + i\beta,$$

$$(3.46) \quad F(u) = \frac{1}{\alpha^2} u^{-2(1+i(\beta/\alpha))}.$$

If

$$(3.47) \quad H_1 = i\beta,$$

$$(3.48) \quad F(u) = \exp[-2\beta iu].$$

Petrov solution corresponds to $H_1 = 0$, whose space is symmetric and $DC_{22} = 0$.

If $\varrho_1 \neq 0$,

$$(3.49) \quad H_2 = \varrho_1 |C_{22}|^{\frac{1}{2}} C_{22}^{-\frac{1}{2}}$$

is a complex invariant. This invariant cannot be constant and $\varrho_1 \neq 0$ at the same time.

We conclude: Homogeneous empty spaces of type N are plane wave solutions with parallel rays, ds^2 is given by (32) with values of H determined by (45) and (47). These spaces admit a G_6 transitive group, which admits a G_3 abelian subgroup with lighttype trajectories [4].

3'4. Type D, II, III. - All these types don't admit homogeneous empty space solutions. Let us sketch the proof.

D-type space is such that

$$(3.50) \quad C = \begin{pmatrix} 0 & B & 0 \\ B & 0 & 0 \\ 0 & 0 & 4B \end{pmatrix}.$$

If local tetrad is determined up to a transformation (3.18) with $\gamma = 0$, B is a scalar invariant.

Because of Bianchi identities given in (3.2) $\sigma^2 = 0$ and the space is of type III, a conclusion which contradicts the hypothesis.

If the empty space is supposed to be of type II, a canonical tetrad can be determined such that

$$(3.51) \quad C_{\alpha\beta} = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{12} & 1 & 0 \\ 0 & 0 & 4C_{12} \end{pmatrix}$$

again Bianchi identities imply that $\sigma^2 = 0$, which is impossible. The type III, admits a canonical tetrad of the form

$$(3.52) \quad C_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C_{23} \\ 0 & C_{23} & 0 \end{pmatrix}.$$

In this case one uses at the same time, Bianchi identities, homogeneity and empty space equations to come to a contradiction.

4. - Empty spaces with harmonic range of the characteristic isotropic vectors.

A very short proof allows to say that no empty space solutions of type I exists with harmonic range for the characteristic isotropic vectors. If this is so the canonical tetrad given in (2.1), can be chosen so that $B = 0$.

From Bianchi identities, given in (3.2), we have

$$(4.1) \quad \sigma^2 \wedge Z^2 = \sigma^1 \wedge Z^1 = 0$$

and by differentiation, and definitions (2.25)

$$(4.2) \quad d\sigma^2 \wedge Z^2 - \sigma^2 \wedge dZ^2 = 0 ,$$

$$(4.3) \quad (\Sigma_1 + \sigma^2 \wedge \sigma^3) \wedge Z^2 - \sigma^2 \wedge (-\sigma^3 \wedge Z^2 + \sigma^2 \wedge Z^3) = 0 ,$$

$$(4.4) \quad \Sigma_1 \wedge Z^2 = AZ^1 \wedge Z^2 = 0 .$$

The space would be flat. Let us also remark that $\sigma^2 \wedge Z^2 = 0$ tells us that the congruence θ^0 is geodesic and shear free, then from the Goldberg-Sachs theorem an empty space solution would be singular [5].

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INTERVENTI E DISCUSSIONI

— J. EHLERS:

I should like to mention that a calculus very similar to the one presented by Prof. Debever has been developed by K. Bichteler, a member of the relativity group at Hamburg University (K. BICHTELER: *Diplomarbeit* (Hamburg University, 1962); published partly in *Zeits. Phys.*, **178**, 488-500 (1964)). Instead of using the (global) isomorphism (2.6) Bichteler bases his formalism on the local isomorphism between $\mathcal{L}_+^\uparrow$ and $SL(2, C)$. Therefore his formulae, though equivalent to those of Prof. Debever *et al.*, are formally even more closely related to the original Newman-Penrose formalism which also uses 2-component spinors. Bichteler has used his calculus to treat very elegantly pure gravitational radiation fields.

— G. C. McVITTIE:

This is a most remarkable piece of work regarded as the solution of a problem in differential geometry. However the title of the paper contains the words « general relativity »; this theory is, for me at least, a branch of mathematical physics which claims to say something about the world. It seems to me that the metrics arrived at having some very odd features. For example, in (3.15),

the coefficient of $(dx^2)^2$ is zero at $x' = \infty$ and infinite at $x' = -\infty$. Thus space-time seems to have effectively only three dimensions in the first case and one might guess that there was a mass-singularity present in the second. These may, of course, be consequences of the use of a particular co-ordinate system. But they do draw attention to the need of a physical interpretation of the metrics. I have the impression that the physical interpretation is a matter of little interest to the investigators, who are essentially pure mathematicians. Is this view correct? If it is not, would the speaker give us the physical interpretation of his results?

— A. LICHNEROWICZ:

Une certaine morale apparaît clairement à la suite de la belle conférence de Debever et de l'intervention de M. Ehlers. Il n'y a aucun avantage spécifique du point de vue *algébrique* à employer un formalisme spinoriel pour des questions proprement tensorielles.

Je reconnais volontiers l'excellence des résultats obtenus en formalisme spinoriel par Penrose et l'école anglaise (certains de ces résultats figurent déjà sous une forme de « Géométrie projective » dans un ancien papier de Ruse sur le tenseur de courbure en dimension 4).

Mais ces résultats ne sont point liés au formalisme. Ce n'est peut-être pas un hasard que ce formalisme ait été particulièrement employé dans des pays qui n'ont pas l'habitude d'utiliser des repères mobiles généraux et de calculer sur des formes, ce qui fournit un formalisme un peu plus court et équivalent: le nombre des indices *éventuels* est diminué de moitié. Il y a un isomorphisme entre les deux points de vue.

— R. DEBEVER:

The present method being, mathematically, strictly equivalent to Bichteler work the choice is only a matter of taste. We have the feeling of being much closer to the usual tensorial methods and of developing the formalism in a more compact form.

I also thank Prof. Lichnerowicz to remember the name of Prof. H. S. Ruse whose papers played a prominent role in the study of the Riemann tensor.

About physical interpretation of the type *I* solution, this is precisely an example which illustrates the fact that Einstein equations of empty space are not sufficient to give us a procedure to describe the physical situation involved, if any. As Prof. H. Taub pointed out, one has to look in the large number of already known solutions for further arguments in this sense.

[Alla discussione parteciparono anche P. G. BERGMANN, W. B. BONNOR, A. H. TAUB.]

Gravitation and Unified Picture of Matter.

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1. - Introduction.

The great Galilean Jubilee offers an opportunity of discussing some new possibilities of the unified picture of the world. It is well known that impossibility of the universal classical mechanical picture became clear under some dramatic circumstances at the beginning of 20th century. The programmes of universal electromagnetic or later of unified geometrical picture proved also to be too narrow in view of quantum treatment of multitude of elementary particles. In these days new interest arose in the building of unified theory of « ordinary » matter, including elementary particles and their excited states: « resonons », but usually gravitation is as yet left aside.

We may point here some possibilities of including gravitation in unified picture of matter and also suggest the necessity of taking into account cosmological phenomena when trying to construct not merely « local » but « natural » picture of the world. J. A. Wheeler in his geometrodynamics aims also at the construction of some « natural » unified picture; in some points *e.g.* in emphasizing importance of mutual transmutations of ordinary matter and gravitational field, this programme is not far from ours.

2. - Nonlinear theory.

Starting from ordinary matter we may draw attention once more to non-linear spinor theory, being a generalization of de Broglie's fusion hypothesis. It is especially important for applications to gravidynamics as in both cases nonlinearity is of genuine, primordial character and not solely induced by vacuum effects. There are many other attempts aiming at unified theory *e.g.* Sakata model, octets, rotator model of de

Broglie-Vigier-Takabayasi and others, developed recently by Yukawa and his collaborators (which partly makes use of our idea of fusing minkowskian and isotopic spin spaces), further also dynamical compensational approach (in spirit of Yang-Mills, Sakurai and others), reggesized dispersional treatment, more formal but powerful group approach. Of course there are many links between all them, *e.g.* between compensation and so succesful SU_3 group.

Not entering here into discussion we present a simple derivation of the mass which is one of chief problems of the whole theory. Suppose the vacuum is degenerate in spirality, due for instance to pairing of particles of opposite spirality. This leads to an energy gap in excitation spectrum, which is considered to correspond to the nucleonic mass. To remove degeneracy add to Lagrangian a mass term

$$(2.1) \quad \mathcal{L}'_0 = \mathcal{L}_0 + m\bar{\Psi}\Psi; \quad \mathcal{L}_{\text{int}} = \mathcal{L}_{\text{int}} - m\bar{\Psi}\Psi.$$

Leaving spirality conservation and introduce anomalous greenians $S_C^{\mathcal{L}R}$ alongside with normal $S_C^{RR}, S_C^{\mathcal{L}\mathcal{L}}$ using perturbation calculation and requiring compensation of diagrams, corresponding to $\mathcal{L}'_{\text{int}}$ with an ingoing φ -line and outgoing χ line, one gets

$$(2.2) \quad -\frac{3}{2}\lambda \text{Sp } S_C^{RL} \mathcal{L}(0) : \varphi^+(x) \chi(x) : = m : \varphi^+(x) \chi(x) :$$

or in impulse space

$$(2.3) \quad m = \frac{3i\lambda m}{(2\pi)^4} \int \frac{d^4p}{p_0^2 - m^2 - i\varepsilon}.$$

Non-trivial solution of this equation happens to coincide up to numerical factor with the result of Nambu and Jona-Lasinio who started from different considerations:

$$(2.4) \quad m^2 \ln \left(1 + \frac{\mathcal{L}^2}{m^2} \right) = \mathcal{L}^2 - \frac{16\pi^2}{3l^2}.$$

Supplementary conditions $\lambda < 0$ (effective attraction), $-\mathcal{L}^2 \lambda \geq 16\pi/3$ were taken into account. Cutting at $\mathcal{L}^2 = m$, we get $ml = 13, 0$; at $\mathcal{L}^2 = 3m^2$, $ml = 5.75$. These results obtained in very simple manner are practically coinciding with conclusions of Heisenberg and collaborators, who used complicated new Tamm-Dancoff method and indefinite Hilbert space metrics. Although we are emphasizing the necessity of introducing new type of propagator for nonlinear equation, one can see that even a simplified quantization leads to qualitatively satisfactory mass spectrum of barions. The gravitons can be constructed like photons and a preli-

minary evaluation in analogy with derivation of fine structure constant leads to reasonable value of gravitational constant. Of course for classification of particles a non-linear generalization accounting also invariance under SU_3 would seem to be promising.

In view of success of SU_3 group one may prefer now to start not from iso-spinors (Heisenberg) but from an unitary spinor $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \psi$, whose 3 components may be considered as « quarks » of Zweig-Gell-Mann and which possess further as components ordinary Weyl or Dirac spinors. We prefer however to take massless quarks and to arrive at masses by the above sketched procedure. Using some relations between bilinear invariants *e.g.*

$$(\bar{\varphi}\varphi)^2 = \frac{4}{3}(\bar{\varphi}\lambda_i\varphi)^2$$

(λ_i -infinitesimal generators of SU_3), we get for nonlinear supplementary Lagrangian

$$\mathcal{L}' = (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2.$$

In this way one may hope to obtain by nonlinear « excitation » (or fusion) all real strongly interacting particles with their old spin etc. and new strange etc. properties.

3. - Tetrad theory of gravitation.

The interaction of fermions with gravitation requires introduction of tetrads, moreover introducing of h_μ^a instead of $g_{\mu\nu}$ as potentials means not a mere retranslation of equations of general relativity as some new supplementary conditions are needed and there can arise new invariants. Compensational treatment of gravitation is based also on tetrads. So one is inclined rather to speak about a kind of tetrad generalization of Einsteinian general relativity.

Let us introduce at each point an affine repere (latin indices). Then for Lamé's coefficients

$$(3.1) \quad h_\mu^a = h_\mu^a; \quad g_{\mu\nu} = h_\mu^a h_{a\nu}; \quad A = \det |h_\mu^a| = \sqrt{-g}.$$

One considers two groups of transformations: *A*) arbitrary transformations of co-ordinates, *B*) local orthogonal transformations of tetrads which connect nonholonomic co-ordinate systems. Combined covariant

derivative reads

$$(3.2) \quad {}^*\nabla_\sigma \mathcal{A}_{(a)}^\mu = \mathcal{A}_{(a),\sigma}^\mu + \Gamma_{\sigma\mu}^\mu \mathcal{A}_{(a)}^\nu - \Delta_{\sigma(ab)} \mathcal{A}^{\mu(b)}.$$

Christoffel symbols, Ricci coefficients are equal

$$(3.3) \quad \Delta_{\sigma,\mu\nu} = C_{\sigma\nu,\mu} - C_{\nu\mu,\sigma} - C_{\sigma\mu,\nu}$$

$$(3.4) \quad C_{\sigma,\mu\nu} = \frac{1}{2} h_{a\sigma} (h_{\nu,\mu}^a - h_{\mu,\nu}^a) = h_{a\sigma} h_{[\nu,\mu]}^a.$$

Variation of scalar curvature

$$(3.5) \quad \begin{aligned} \Delta R &= 2(\Delta A_{a..}^{aa})_{,\sigma} + \Delta(A_{a,bc} A^{b,ca} - A_{a,b}^a \Delta_{c..}^{bc}) = \\ &= 4(\Delta C_{a,..}^{aa})_{,\sigma} + \Delta(4C_{a,b}^a C_{c..}^{cb} - \Delta_{a,bc} C^{a,bc}) \end{aligned}$$

over h_μ^a yields Einstein equations in tetrad form

$$(3.6) \quad R_{na}{}^{n\mu} - \frac{1}{2} h_\mu^a R = -\kappa T_a^\mu.$$

As a possible supplementary condition we choose with V. I. Rodičev the « quasi-harmonic » one

$$(3.7) \quad \Delta_{na}{}^n = \frac{1}{\Delta} (\Delta h_a^\sigma)_{,\sigma} = 0$$

with guarantees the continuity of nonholonomic orthogonal system of co-ordinates and goes over in known harmonic de Donder condition for holonomic reperes. With this quasi-harmonic condition we get

$$(3.8) \quad R = -\Delta_{abc} C^{abc}$$

which is scalar both for groups A and B satisfying our quasi-harmonicity. Then field equations read

$$(3.9) \quad \Delta_m{}^{\sigma a}{}_{ig} = \bar{t}_m^\sigma + \kappa T_m^\sigma$$

where $\bar{t}_m^\sigma$ characterizes energy of the gravitational field.

In contrast to general relativity which locally does not distinguish 1) fictitious field, due to nonstationary co-ordinates, 2) inertial field due to noninertial reference system, 3) gravitational field caused by distribution of masses, now the tetrad formalism suggests a new possible interpretation of inertia.

Let us consider now the problem of energy, which as well known up to now could not find any satisfactory generally recognized solution. Applying Noether's theorem to complete or abridged Lagrangians one got either Møller-Mitskevič or Einstein expressions for the energy complex; Lorentz complex obtained by variation over $g_{\mu\nu}$ yielded vanishing result. Now one clearly must apply Noether formalism not to $g_{\mu\nu}$ but to h_μ^a and gets

$$(3.10) \quad \begin{cases} \bar{\mathcal{L}}_0 = \frac{1}{2} \Lambda G, \\ G = \Delta_{a,bc} \Delta^{b,ca} - \Delta_{a,}{}^{ab} \Delta_{c,}{}^{bc} \\ \bar{t}_m^\sigma = -\frac{1}{2} G h_m^\sigma + 2 \Delta^{a,b\sigma} C_{a,bm} + 4 C_{a,}{}^{aa} C_{,bm}^b - 4 C_{a,}{}^{ba} C_{,bm}^\sigma. \end{cases}$$

On the other hand from complete Lagrangian

$$(3.11) \quad \mathcal{L}_0 = \frac{1}{2} \Lambda R$$

we get another covariant quantity

$$(3.12) \quad t_m^\sigma = -\frac{1}{2} R_m^\sigma - 2 \Delta^{a,b\sigma} C_{a,bm} + 2 C^{a,ab} C_{a,bm} - \\ - 4 C_{,ab}^a C^{(a,b)}_m + 2 g^{a\sigma} C_{,am,q}^a - 2 h_a^{(\sigma} h^{a)}_b C_{m,q}^{a \cdot b}.$$

We may remark that both energy complexes can be obtained also by usual procedure of separating a divergence in field equations and shifting remaining terms on the right hand side.

Both energy expressions can be obtained by means of superpotentials

$$(3.13) \quad \begin{cases} \Lambda(\bar{t}_m^\sigma + \kappa T_m^\sigma) = \frac{\partial}{\partial x^q} (\Lambda \Delta_{m,}{}^{\sigma q} + 2 \Lambda \Delta_{a,}{}^{a(\sigma} h^{q)}_m), \\ \Lambda(t_m^\sigma + \kappa T_m^\sigma) = \frac{\partial}{\partial x^q} (\Lambda C_{m,}{}^{\sigma q}). \end{cases}$$

Using from beginning the quasi-harmonic condition we get by means of field equations an expression

$$(3.14) \quad \bar{t}_m^\sigma = \frac{1}{2} \Delta_{a,bc} C^{a,bc} h_m^\sigma + 2 \Delta^{a,b\sigma} C_{a,bm}$$

which results also from $\bar{t}_a^\sigma$ by applying this condition.

The total energy-impulse

$$(3.15) \quad P_m = \frac{1}{\kappa c} \int (d^3x)_\sigma (At_m^\sigma + AT_m^\sigma) = \frac{1}{\kappa c} \oint (dx^2)_{\sigma_0} (AC_m^{\sigma_0})$$

is an affine vector in respect to Lorentz transformations but a scalar in respect to arbitrary transformations of co-ordinates.

Applying Noether formalism to local 4-rotations we get angular momentum of the field, which with quasi-harmonic condition reads

$$(3.16) \quad \Sigma_{mn}^\sigma = \frac{2}{\kappa c} C_{mn}^\sigma$$

which explains the meaning of C_{mn}^σ . We see that tetrad formalism led already to some interesting conclusions in gravodynamics and its development seems rather promising removing in particular serious difficulties in definition of energy, well known from all previous investigations.

4. - Non-linearity and torsion.

At this stage one can say that the physical reality seems to manifest itself in dualistic manner, being described by means of a some nonlinear spinor (ordinary matter) and on the other side by a tetrad gravitational potential which also can be put in spinor form (which suggests introducing of a kind of combined universal spinor).

On the other side geometrodynamics prefers to operate with bosonic geometrical fields and in this connection J. A. Wheeler, recognizing the interest of nonlinear spinor theory, pointed the absence of geometrical meaning of nonlinear term, which moreover possessed in his opinion only local character.

But an attention must be paid to Rodičev's theorem which just tries to endow the nonlinearity with geometrical meaning. Let us use a general affine connection ${}^*\Gamma_{\alpha\lambda}^\mu$ which is nonsymmetrical in lower indices. In simplest case when torsion tensor is fully antisymmetric (Galilean metrics, geodesics are straight lines) we have

$$(4.1) \quad {}^*\Gamma_{\sigma\mu,\nu} = K_{\sigma\mu,\nu} = \Phi_{[\sigma\mu\nu]}$$

and curvature scalar

$$(4.2) \quad {}^*R = \Phi_{[\sigma\alpha\beta]} \Phi_{[\sigma\beta\alpha]}.$$

For spinors in spaces endowed with torsion we get

$$(4.3) \quad \Psi_{;\sigma} = \Psi_{;\sigma} - *B_{\sigma} \Psi$$

$$(4.4) \quad *B_{\sigma} = \frac{1}{4} * \Delta_{\sigma,ab} \gamma^a \gamma^b + iI \varphi_{\sigma}$$

generalizing well known coefficient of curved space. In our simple case

$$(4.5) \quad * \Delta_{\sigma,\mu\nu} = \Phi_{[\sigma\mu\nu]}.$$

Using spinor Lagrangian and passing to pseudo-vectors we get for action

$$(4.6) \quad T = \int \{ \mathcal{L} - b * R \} (dx)$$

$$(4.7) \quad I = \int \left\{ \frac{1}{2i} (\Psi^+ \gamma_{\alpha} \Psi_{;\alpha} - \Psi^+_{;\alpha} \gamma_{\alpha} \Psi) - (\Psi^+ \gamma_{\alpha} \Psi) \varphi_{\alpha} - \right. \\ \left. - (\Psi^+ \gamma_{\alpha} \gamma_5 \Psi) \tilde{\varphi}_{\alpha} + \frac{1}{2\lambda_0^2} \tilde{\varphi}_{\alpha}^2 \right\} (dx).$$

Putting for simplicity electromagnetic potential $\varphi_{\lambda} = 0$ one gets after variation over $\tilde{\varphi}_{\lambda}$ and Ψ^+

$$\gamma_{\alpha} \Psi_{;\alpha} + i\lambda_0^2 (\Psi^+ \gamma_{\alpha} \gamma_5 \Psi) \gamma_{\alpha} \gamma_5 \Psi = 0,$$

i.e., the fundamental nonlinear spinor equation (but as yet without eventual SU_3 or other analogous refinements) moreover just with pseudo-vectorial term chosen by Heisenberg from the point of view of group properties among various nonlinearities pointed by us. Anyhow a more systematic study of torsion is suggested by this result even for the better analysis of the ordinary curved space, where Riemannian connection can be divided in two mutually compensating parts both endowed with torsion.

5. - Gravitational transmutations of elementary particles.

Gravidynamics in tetrad form can be quantized along the same lines as ordinary form based on $g_{\mu\nu}$ and leads to canonical and covariant commutation relations between h_{ν}^{α} and Ricci's coefficients plying the role of momenta *e.g.*

$$(5.1) \quad (h_{\alpha}(\varepsilon) \Delta^{0*}(\varepsilon) - \Delta(\varepsilon)^{0*} h_{\alpha}(\varepsilon)) = \frac{1}{2} \hbar.$$

One can establish also relations of the type of «individual» errors discussing some «Gedankenexperimenten» in the spirit of previous considerations (Bronstein, Regge, Treder, de Witt).

Without entering into these developments, we may apply the results of quantization in linear approximation to some important processes above all to calculation of probability of transmutation of a pair of fermions in 2 gravitons. The possibility of such transmutations suggested by us was discussed later by many authors (A. A. Sokolov, I. Piir, J. A. Wheeler, D. Brill, M. Korkina, J. S. Vladimirov, J. Feynman, J. Weber). There arise in gravidynamics some supplementary diagrams due to nonlinearity, when comparing with electrodynamics.

In nonrelativistic limit $E^2 \sim m^2 \gg p^2$

$$(5.2) \quad d\sigma = \frac{m^2 \kappa^2}{64(8\pi)^2} \frac{C}{V} d\Omega.$$

Applying qualitatively such calculations also to high energies we obtain

$$(5.3) \quad d\sigma = \frac{\kappa^2 E^2}{128(8\pi)^2} (3 \sin^2 2\theta + 2 \sin^4 \theta) d\Omega.$$

We remark that cross-section of photon-graviton annihilation tends to constant limit (at such extrapolation)

$$(5.4) \quad \sigma \sim 10^{-68} \text{ sm}^2$$

at the energies of the order

$$E \sim \sqrt{\frac{r_e}{r_g}} mc^2$$

the cross-section of photon graviton, two-graviton and two-photon annihilation of a pair of particles are of the same order.

In view of the some renewed interest in torsion we have quantized this field, putting the torsion tensor in the form:

$$(5.5) \quad S_{\alpha\beta\gamma} = \frac{1}{2} \left(\frac{\partial \varphi_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial \varphi_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial \varphi_{\gamma\alpha}}{\partial x^\beta} \right)$$

getting the corresponding quanta: «torsions» and investigating their interaction with spinor and other fields.

6. - Compensational treatment of gravitation.

We draw attention to the fact that gravitational field can be naturally connected with other fields allowing « compensational » treatment and realizing dynamically some conservation laws. Requiring invariance under some group *e.g.*, of isotopic rotations, but with parameters depending from co-ordinates, one must introduce a compensating field to restore invariance (Yang-Mills, Sakurai and others). In the same sense invariance under localized gauge group

$$(6.1) \quad \Psi \rightarrow \Psi \exp [ie\omega(x)]$$

required introduction of electromagnetic vector potential

$$(6.2) \quad \mathcal{A}_\mu \rightarrow \mathcal{A}_\mu + \frac{\partial \omega(x)}{\partial x^\mu}.$$

It seems that some predicted in this manner particles were recently discovered as « resonons ». The whole method seems to bring new arguments in favour of more narrow connection of Minkowskian and isotopic spin spaces (external and internal degrees of freedom) (suggestion of Pais, Brodski, Sokolik and ourselves, Yukawa developed by de Broglie-Vigier *et al.*).

Without entering into instructive history of compensational treatment of gravitation (Utiyama, Brodski-Ivanenko-Sokolik-Frolov, Kibble, Schwinger) we may consider the full inhomogeneous Poincaré-Lorentz group. For an arbitrary group Γ connected with co-ordinate transformations, the compensating derivative has the form

$$(6.3) \quad Q_{|a}^{\mathcal{A}} = h_a^\sigma Q_{,\sigma}^{\mathcal{A}} - I_{Bm}^{\mathcal{A}} \mathcal{A}_a^m Q^B$$

$I_{Bm}^{\mathcal{A}}$ -generator of that representation of Γ under which is transformed the field $Q^{\mathcal{A}}$. In general case one needs for compensation two independent compensation fields $\mathcal{A}_a^m$ and h_a^σ which will be connected by means of equations of motion.

$\mathcal{A}_a^m$ compensates terms arising at local transformations of $Q^{\mathcal{A}}$ and the field h_a^σ compensates term due to local transformations of coordinates under Γ .

One constructs locally invariant Lagrangian by multiplying with $\Delta = \det |h_a^\sigma|$ and extending the ordinary derivative by compensating one

$$(6.4) \quad \mathcal{L}_{(Q)} = \Delta L_{(Q)}(Q^{\mathcal{A}}, Q_{|a}^{\mathcal{A}}).$$

For Lagrangian of the compensating field itself we have

$$(6.5) \quad \mathcal{L}_0 = AL_0(\mathcal{F}_{ab}{}^m)$$

with

$$(6.6) \quad \mathcal{F}_{ab}{}^m = h_a^\nu \mathcal{A}_b{}^m{}_{,\nu} - h_b^\nu \mathcal{A}_a{}^m{}_{,\nu} + \mathcal{A}_c{}^m (h_a^\sigma h_b^\tau h_\sigma^c - h_{b,\sigma}^\tau h_a^\sigma h_\tau^c) + C_{q,n}^m \mathcal{A}_a{}^q \mathcal{A}_b{}^n.$$

One gets the field equations by variation over potentials h_a^σ and $\mathcal{A}_\sigma^m$

$$(6.7) \quad \frac{\delta \mathcal{L}_0}{\delta \mathcal{A}_\sigma^m} = -\kappa \frac{\delta \mathcal{L}_{(Q)}}{\delta \mathcal{A}_\sigma^m}$$

$$(6.8) \quad \frac{\delta \mathcal{L}_0}{\delta h_\sigma^a} = -\kappa \frac{\delta \mathcal{L}_{(Q)}}{\delta h_\sigma^a}$$

with our well known coefficients Γ : (Fock-Ivanenko).

Hence

$$(6.9) \quad \frac{\partial}{\partial x^\sigma} \frac{\partial \mathcal{L}_0}{\partial \mathcal{F}_{\sigma\mu}{}^m} = A \mathcal{F}_0{}^\mu{}_m = A \mathcal{F}_0{}^\mu{}_m + \kappa A \mathcal{F}_{(Q)}^\mu{}_m = \left(-\frac{1}{2} \frac{\partial \mathcal{L}_0}{\partial \mathcal{A}_\mu^m} - \frac{\kappa}{2} \frac{\partial \mathcal{L}_{(Q)}}{\partial \mathcal{A}_\mu^m} \right).$$

$$(6.10) \quad \mathcal{F}_{\mu\nu}{}^m \frac{\partial \mathcal{L}_0}{\partial \mathcal{F}_{\mu\nu}{}^m} - \frac{1}{2} h_\mu^a \mathcal{L}_0 = -\kappa A T_\mu^a = -h_\mu^a \mathcal{L}_{(Q)} + \frac{\partial \mathcal{L}_{(Q)}}{\partial Q^\sigma{}_{/a}} Q^\sigma{}_{;\mu}.$$

In the case of inhomogeneous Lorentz group h_a^ν are Lamé's reperes, $\mathcal{A}_\sigma^{ij}$ -Ricci's coefficients. Putting in (6.3) the generators of corresponding representations of the Lorentz-group we get covariant (compensating) derivatives for vectors etc. For spinor one gets in this manner a new derivation

$$(6.11) \quad \Psi_{|a} = h_a^\sigma \Psi_{,\sigma} - \Gamma_a \Psi$$

with our well known coefficients Γ_a (Fock-Ivanenko).

For Lagrangian

$$(6.12) \quad \mathcal{L}_0 = \frac{1}{2} A R_{ij} R^{ij}$$

one arrives with (6.10) at tetradic form of Einstein equations, and (6.9) gives an expression of torsion tensor by means of the spin of the external field

$$(6.13) \quad K_{ab}^\mu = -\frac{\kappa}{2} (S_{ab}^\mu + h_{[a}^\mu S_{b]\sigma}^\sigma).$$

For longitudinal part of the compensating field we have

$$(6.14) \quad \mathcal{A}_a^{(l)m} = -(1/\alpha) h_a^\alpha \tilde{\mathcal{U}}_c^\alpha I_{\mathcal{A}}^{cm} \mathcal{U}_{\alpha,a}$$

α -some number depending from a given representation of $\mathcal{U}_\alpha^\alpha$. In the case of Lorentz group one naturally takes tetrads h_a^α for $\mathcal{U}$.

Although the development of compensational treatment of gravitation is by no means finished and raises many problems, for instance of eventual generalization for torsion, we see that the basic theory is powerful enough to give an important reinterpretation of ordinary General Relativity and to contribute to its present day tetradic generalization.

7. - Cosmological remarks.

Aiming at a construction of some kind of « natural » unified picture of the world and not only of local theory one must carefully take into consideration all possible links between cosmology and elementary particles.

In this connection starting from well known empirical fact that

$$(7.1) \quad \frac{m_e}{\alpha_e} = \frac{m_N}{\alpha_g} = a \approx 1,4 \cdot 10^{-25} \text{ g}$$

(m_e , m_N -masses of electron and nucleon, α corresponding fine structure constants) one may try to proceed further and build with D. Kurdgelaidze corresponding masses for the case of Fermi interaction ($m_F = a\alpha_F \approx 10^{-40} \text{ g}$) and for the gravitation

$$(7.2) \quad m_g = a\alpha_g \approx 6 \cdot 10^{-66} \text{ g}.$$

If one supposes gravitational field in some manner endowed with such mass one can compare the wave equation

$$(7.3) \quad \left[\square - \left(\frac{m_g c}{\hbar} \right)^2 \right] \eta_{\mu\nu} = 0$$

with linearized form of Einstein equations supplemented by cosmological term λ_0 . One gets then for Hubble constant

$$(7.4) \quad h_1 = \lambda_0^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \frac{m_g c^2}{\hbar} = \frac{\kappa a^2 c}{\sqrt{2} \hbar} \simeq 4 \cdot 10^{-18} \text{ sec}^{-1},$$

in good agreement with observational value (cf. also Gomide).

We may draw attention further to a tempting possibility to connect the observed overwhelming concentration of particles in our part of universe with its expansion, guessing that for contracted model one must pass to anti-particles. Among further as yet scarce links between cosmology and microphysics one must keep in mind also interesting considerations about the arrow of time (Bondi, Hoyle, Hogarth) and Wheeler and Hoehn-Dehnen treatment of boundary Machian conditions.

We see that both nonlinear theory, compensational viewpoint and mutual transmutations all seem to be promising in establishing fundamental new relations between ordinary matter and gravitation, preliminary considerations point even on the links with cosmology.

May be an unified picture of ordinary matter will even prove to be impossible without account of gravitation and cosmological features?

Note added in proofs.

Developping these ideas we may suggest that T -parity and combined CP -parity would be in general not conserved (if Lüders theorem holds in gravidynamics), the expansion acting, as a kind of effective very weak cosmological fore, differently on particles and anti-particles. This conclusion can be contrasted with recent discovery by Fith-Cronin and their collaborators of the apparent violation of combined parity at K_2 mesons decay. We believe with D. KURDGELADZE that the recent interesting hypothesis of CABIBBO, LEE *et al.* to explain this anomaly by introducing a new field produced by the hypercharge of our Galaxy, anyhow must be supplemented by cosmological viewpoint as otherwise the global effect of all galaxies taken as static not expanding distribution would lead to infinity analogously to well known paradoxon of Newtonian static cosmology.

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INTERVENTI E DISCUSSIONI

— J. TREDER:

The question « What the gravitational potentials are?—the 10 $g_{\mu\nu}$ or the 16 components of the tetrads? » is very important for general relativity. If the 16 tetrad-components are going in the gravitation-theory, free rotations of the tetrads are not possible. The relative orientation of the tetrads are fixed by the physical conditions. Weyl has pointed out that no physical meaningful causes existed for this fixation. But for the theory of the interactions between gravitational fields and Fermi fields, the introduction of tetrads is necessary (*). But, if we have no causes of the meaning that only the $g_{\mu\nu}$ have a physical meaning, the Einstein-Lagrangian is only a degenerate combination of the 3 Lagrangians which are giving second-order-field-equations for the « gravitational potential » h_{α}^{α} . These three Lagrangian are the Lagrangians of Weitzenböck (Einstein and Mayer have worked with these Lagrangians). Generally, the Weitzenböck Lagrangians are giving 16 field equations. Only the Einsteinian combination is giving only 10 equations for the 10 $g_{\mu\nu}$.

— A. LICHTNEROWICZ:

Je crois que nous tombons dans le qui-pro-quo parce que deux choses différentes ne sont pas suffisamment distinguées.

1) L'introduction de repères orthonormés (ou tetrades) en un point est nécessaire. Ces repères donnent l'interprétation de grandeur physique, en un point de l'espace-temps relativement aux directions d'espace et de temps qu'ils définissent. Les coordonnées ne servent qu'à la cartographie. Ces repères sont utiles pour une bonne définition des objets spinoriels (*).

2) L'introduction de champs privilégiés de tetrades est une chose différente, grave et sur laquelle il convient de réfléchir. Personnellement, je n'y suis pas favorable pour beaucoup de raisons. En fait la théorie correspondante est fort éloignée de la relativité générale. Elle est trop riche; elle est une particularisation et pas une généralisation.

— H. BONDI:

Do I correctly understand your attitude to tetrads if I say that two spaces

(*) *Remark of Ivanenko*: as shown by A. FOCK-D. IVANENKO and H. WEYL.

with the same metric, but different tetrad systems are regarded as physically distinct by you, but are regarded as physically identical by several others who also use tetrads? And how does the difference show up?

Furthermore, does the greater fixing of the space at infinity mean that your theory comes closer to explaining the definition of inertial frames in terms of boundary conditions at infinity (Mach's principle) than ordinary general relativity?

— P. G. BERGMANN:

1) Formally, one can introduce spinors without tetrads; but the spinors require the unimodular group, which is locally isomorphic to the Lorentz group. Hence, in the presence of Fermian fields, the introduction of tetrads is a matter of convenience, not of principle. Incidentally, I agree with the remarks by Prof. Lichnerowicz.

2) I should like to request further elucidation of your reasons for permitting nonsymmetric affine connections, in view of the considerable formal complications.

— J. A. WHEELER:

The discussion of Professor Ivanenko is useful in reminding us of many important questions on which physics has yet to take a final position. Among these issues it is difficult to name any one which is more important than this: how can the existence of *spin* in nature be reconciled with Einstein's long term dream for a purely *geometrical* description of all of physics? We know very well that there is no such thing as spin $\frac{1}{2}$ in Einstein's standard theory of relativity. Gravitational waves have spin two. Electromagnetism can be described in Einstein's theory as an aspect of geometry (Rainich and Misner), and electromagnetic waves have spin 1. But there is no spin $\frac{1}{2}$. To be sure, one can introduce tetrads, as a matter of convenience in describing the geometry. Tetrads also help in describing a spinor field when such a field is introduced as a foreign entity moving about in the spacetime geometry. But the use of tetrads in formulating Einstein-Maxwell theory in no way changes the conviction that theory deals solely with fields of integral spin. It is completely incapable of explaining the presence of spin $\frac{1}{2}$ in nature whether formulated with or without tetrads. Therefore some fundamental change is required in the theory if it is to embrace spin $\frac{1}{2}$. One suggestion about the direction for such a change is suggested by recent results of de Witt, kindly communicated in a letter of 19 June 1964 and a personal discussion of 8 August 1964. In brief, he shows that the wave equation for the state function of gravitation theory, $\psi = \psi^{(3)}(\mathcal{G})$, has points of similarity to the Klein-Gordon wave equation for the state function of a particle, $\psi = \psi(x)$. In both cases the differential equation is of second order. In the case of the Klein-Gordon equation Dirac « took the square root » to arrive at a first order wave equation.

This equation is *not* equivalent to the Klein-Gordon equation but it is compatible with experience. Similarly de Witt suggests it will be interesting to « take the square root » in the sense of Dirac of the functional wave equation

to which he has arrived. So much for the idea in broad outline the details are summarized in Table:

TABLE - *Analogy between the Klein-Gordon equation and the de Witt equation.*

	Klein-Gordon	de Witt
System under consideration	Particle	Geometry
Configuration described by	x	${}^{(3)}\mathcal{G}$
Key equation in classical theory	$p_\mu p^\mu + m^2 = 0$	$(\text{Tr } k)^2 - \text{Tr } k^2 + {}^{(3)}R = 0$
Quantum wave equation	$\square \psi + m^2 \psi = 0$	See below
« Square root »	$\Gamma^\mu \partial_\mu \psi + m\psi = 0$	To be written out by analogy

The de Witt equation has the form:

$$g^{-\frac{1}{2}}(\delta/\delta g^{(ij)})g^{\frac{1}{2}}g^{(ij)KLB}(\delta/\delta g^{(kl)})g^{-\frac{1}{2}}\varphi - \lambda^{\frac{1}{2}(3)}R\psi({}^{(3)}\mathcal{G}) = 0.$$

To take the square root of this equation in this sense of Dirac will lead, not to a spinor *particle* of mass m , as in the case of Klein-Gordon equation, but to a spinor *field* of zero rest mass. It is of great interest and importance to know whether this spinor field has any correspondence with what we know of neutrinos.

— C. MÖLLER:

I should perhaps make a few remarks about the difference between Ivanenko's point of view and my present view regarding the use of tetrads in general relativity. Up to the fall of 1963 I tried to find an expression for the distribution of the energy in gravitational fields and for this I found it necessary to regard the tetrads as true field variables which should be determined uniquely up to a constant Lorentz rotation. Since Einstein's field equations determine the metric, only, they had to be supplemented by six further covariant equations. In view of the arbitrariness involved in the setting up of these supplementary conditions and also because it seemed so difficult to imagine how one could measure the gravitational energy content in a small part of space I came to the conviction that Einstein's old point was right and that only the *total* energy and momentum of an insular system can be regarded as a measurable quantity. From this point of view we do not need to fix the tetrads completely and the problem of setting up supplementary equations does not arise. Only in the case of a completely empty space one has to assume that the tetrads form a stationary vector field in order to avoid the Bauer difficulty which is the most disagreeable feature of Einstein's energy-momentum complex. Moreover, in the case of an insular system one has to assume certain boundary conditions for the tetrads at spatial infinity. Apart from that we can freely rotate the tetrads independently throughout space-time and it can be shown that the resulting total four-momentum P_i as well as the asymptotic expression for the complex T_i^h at large spatial distances are invariant under these rotations

of the tetrads. In the case of the Bondi and Sachs solutions the expression for the total energy is in accordance with Bondi's definition of the total mass and moreover, for a nonradiative system, P_i can be shown to be a free 4-vector under arbitrary space-time transformations. From this point of view the tetrads play the role of auxiliary variables similarly as the electromagnetic potentials in electrodynamics and the Lorentz rotations of the tetrads are somewhat analogous to the usual gauge transformations. Maybe I have resigned too early and it is not impossible that the point of view of Ivanenko, which tries to connect gravitational phenomena with others physical phenomena, at the end will turn out to be more fruitful.

— D. IVANENKO:

On the whole we had a very stimulating discussion with many interesting remarks, but I may concentrate here only on some few questions. Although different authors treating spinors in general relativity use such expressions as: convenient, helpful, useful etc., it is important to stress that the description of gravitational interaction of fermions requires introduction of tetrads quite compulsory (Fock-Ivanenko, Weyl). This was emphasized recently also by Ch. Møller, A. Lichnerowicz, H. Treder, J. A. Wheeler and others. So for a spinor tetrad components represent gravitational potentials and may be considered as a field. It seems therefore that remarks of P. Bergmann which run counter this general opinion are connected with some misunderstanding. Above this tetrads are connected in natural way with powerful method of compensating fields (Jang-Mills) and seem to be able to lead to a reasonable energy expression (Møller, Rodičev) perhaps contributing also to further clarification of inertia notion. Whether one prefers to speak in this connection about a «rounding» or a «revision» or a kind of tetradic «generalization» of Einsteinian theory based solely on $g_{\mu\nu}$ metrical and gravitational potentials, is a matter of personal taste.

At present state of the tetrad theory of gravitation many important questions need further investigation as was rightly pointed here by H. Bondi, A. Lichnerowicz, G. Wataghin, M. A. Tonnelat. Not only the best choice of supplementary conditions (old Møller?, Rodičev, Schwinger-Deser?) is not settled, but Prof. Ch. Møller even tentatively proposes now to dispense with such conditions (using instead some boundary conditions). We try to select Rodičev's quasi-harmonic conditions, expressing the conservation of the 4-volume corresponding to some given configuration of events somewhat analogously to Liouville's theorem. Then only 4 tetradic components are fixed, the other 2 being determined by the nature of a given concrete problem. In this respect we are more inclined to Møller's older view point or Schwinger-Deser proposals where some supplementary conditions were used, which restricted to some extent the free rotations of tetrads at different point. Anyhow the various results already obtained on a difficult road of classical and quantum tetradic theory of gravitation seem to permit a rather optimistic attitude.

As to P. Bergmann's question about nonsymmetrical part of connection coefficients, which correspond to torsion I wish to point that torsion or particularly distant parallelism really appear in tetrad theory, also in compen-

sational treatment, in a natural way, though it is not necessary. Anyhow one is led to account explicitly the absence of torsion as does *e.g.* Ch. Møller in his report. Also we may draw once more attention to Rodičev's theorem showing that parallel displacement of spinors in a space endowed with torsion leads to a nonlinear term in Dirac's equation (cf. also R. Finkelstein, A. Peres, Sciama).

In view of importance of some kind of nonlinear spinor equation in the unified theory of matter this result must be considered with some attention.

Perhaps one can even guess proceeding in this way some preliminary link between unified theories of nonlinear or fusionist spinor type and geometrized theories of J. A. Wheeler.

[Alla discussione partecipò anche G. WATAGHIN.]

A New Class of Vacuum Solutions of the Einstein Field Equations. (*)

R. P. KERR and A. SCHILD

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1. - Introduction.

In this paper the general solution of Einstein's empty space field equations, $R_{\mu\nu}=0$, is obtained for a space where the metric has the form

$$(1.1) \quad g_{\mu\nu} = \eta_{\mu\nu} + l_\mu l_\nu .$$

Here $\eta_{\mu\nu}$ is the metric of Minkowski space in co-ordinates which are Cartesian but not necessarily rectangular, *i.e.*, $\eta_{\mu\nu}$ are constants, with signature $+++ -$, and l_μ is null:

$$(1.2) \quad g^{\mu\nu} l_\mu l_\nu = 0 .$$

The reason for considering vacuum solutions of the form (1.1) is that the contravariant components of the metric are easily expressed in terms of the covariant components. In fact,

$$(1.3) \quad g^{\mu\nu} = \eta^{\mu\nu} - l^\mu l^\nu ,$$

where

$$(1.4) \quad l^\mu = g^{\mu\nu} l_\nu = \eta^{\mu\nu} l_\nu ,$$

and the determinant

$$(1.5) \quad (-g) = -\det(g_{\mu\nu}) = 1 .$$

(*) This research has been supported by the Aerospace Research Laboratory, Office of Aerospace Research, and the Office of Scientific Research, U. S. Air Force.

It follows that if l_μ is null with respect to one of the two metrics $g_{\mu\nu}$ and $\eta_{\mu\nu}$, then it is null with respect to the other, *i.e.*, eq. (1.2) implies

$$(1.6) \quad \eta^{\mu\nu} l_\mu l_\nu = 0,$$

and vice versa.

The new vacuum solutions have the following properties:

a) They include as special cases the Schwarzschild solution and the exterior solution of a rotating body which was recently discovered by one of us [1].

b) All vacuum solutions of the form (1.1) are algebraically degenerate in the sense of the Petrov-Pirani classification, l_μ being a multiple Debever-Penrose vector and thus geodesic and shear-free.

c) All vacuum solutions of the form (1.1) admit a one parameter group of motions. The Killing vector K^μ is at the same time a Killing vector of the flat Minkowski metric $\eta_{\mu\nu}$. In fact, with respect to $\eta_{\mu\nu}$, the motion is just a translation along a direction which may be time-like, space-like, or null in the Minkowski space. There are thus three cases:

$$(1.7) \quad \begin{aligned} \eta_{\mu\nu} K^\mu K^\nu &< 0, & \text{(Case I)} \\ &> 0, & \text{(Case II)} \\ &= 0. & \text{(Case III)} \end{aligned}$$

d) In each of the three cases, the general solution is determined by one arbitrary analytic function of one complex variable.

e) If $g_{\mu\nu}$ admits a Killing vector other than K^μ , it also must be a Killing vector with respect to $\eta_{\mu\nu}$.

f) There are at most two essentially different ways of representing a vacuum metric in the form (1.1). For case I, apart from the special case mentioned in *a)*, the representation (1.1) is, in fact, unique, so that the Riemannian space $g_{\mu\nu}$ determines uniquely the null vector field l_μ and the Minkowski space $\eta_{\mu\nu}$. Similar statements hold for cases II and III.

Together with their graduate student, Mr. George Debney, the authors have examined solutions of the (non-vacuum) Einstein-Maxwell equations where the metric has the form (1.1). Most of the results mentioned above apply to this more general case. This work is continuing.

In the following sections, the derivation is outlined of the most important of the above properties, *i.e.*, properties *a)* to *d)*. The full details of all the proofs will be published elsewhere.

2. - Outline of derivation.

A simple direct calculation of the Christoffel symbols $\left\{ \begin{smallmatrix} \varrho \\ \mu \nu \end{smallmatrix} \right\}$ for the metric (1.1), (1.3) and of $\left\{ \begin{smallmatrix} \nu \\ \mu \nu \end{smallmatrix} \right\}$, $\left\{ \begin{smallmatrix} \varrho \\ \mu \nu \end{smallmatrix} \right\} l^\nu$, $\left\{ \begin{smallmatrix} \varrho \\ \mu \nu \end{smallmatrix} \right\} l^\nu l^\mu$ (*), and substitution into the vacuum field equation

$$(2.1) \quad R_{\mu\nu} l^\mu l^\nu = 0 ,$$

yields

$$(2.2) \quad l_{\mu;\nu} l^\nu l^\mu l^\varrho = 0 .$$

By differentiating eq. (1.6), we also have

$$(2.3) \quad l_{\mu;\nu} l^\nu l^\mu = 0 .$$

Thus, in the Minkowski metric $\eta_{\mu\nu}$, the vector $l^\mu_{;\nu} l^\nu$ is null and orthogonal to the null vector l^μ . In a four dimensional space with the signature $3+1$ of space-time, a vector ν^μ orthogonal to a null vector l^μ is either space-like or else a multiple of l^μ . Therefore

$$(2.4) \quad l^\mu_{;\nu} l^\nu = \mu l^\mu .$$

Also, it is easily shown that

$$(2.5) \quad l^\mu_{; \nu} l^\nu = l^\mu_{; \nu} l^\nu = \mu l^\mu .$$

Thus the field eq. (2.1) implies that the null field l^μ is geodesic. We can therefore define a new null vector field $k^\mu = l^\mu / (2H)^{\frac{1}{2}}$, so that

$$(2.6) \quad k_\mu k^\mu = 0 , \quad k^\mu_{; \nu} k^\nu = 0 ,$$

and

$$(2.7) \quad \begin{cases} g_{\mu\nu} = \eta_{\mu\nu} + 2H k_\mu k_\nu , \\ g^{\mu\nu} = \eta^{\mu\nu} - 2H k^\mu k^\nu , \\ k^\mu = g^{\mu\nu} k_\nu = \eta^{\mu\nu} k_\nu . \end{cases}$$

Another simple calculation gives

$$(2.8) \quad R^\sigma_{\varrho\mu\nu} k^\sigma k^\nu = -\ddot{H} k^\sigma k_\sigma ,$$

(*) A comma denotes partial differentiation, e.g., $l^\mu_{;\varrho} = \partial l^\mu / \partial x^\varrho$, a semi-colon, e.g., $l^\mu_{;\nu}$ denotes covariant differentiation with respect to the metric $g_{\mu\nu}$.

where the dot denotes differentiation in the null direction k^μ :

$$(2.9) \quad \dot{H} = H_\mu k^\mu .$$

A Riemannian space-time is called algebraically degenerate (or algebraically special), in the sense of the Petrov-Pirani classification [2], if and only if there exists a null vector field k^μ which obeys the equations

$$(2.10) \quad k_{[\alpha} C_{\sigma] \rho \mu \nu} k^\sigma k^\nu = 0 ,$$

where $C_{\sigma \rho \mu \nu}$ is Weyl's conformal curvature tensor. In vacuum, where the field equations $R_{\mu \nu} = 0$ hold, the Weyl tensor and the Riemann tensor coincide, i.e., $C_{\sigma \rho \mu \nu} = R_{\sigma \rho \mu \nu}$. Thus the condition (2.10) reduces to

$$(2.11) \quad k_{[\alpha} R_{\sigma] \rho \mu \nu} k^\sigma k^\nu = 0 .$$

By (2.8), this is clearly satisfied. Thus all vacuum solutions of the form (1.1), or equivalently (2.7), are algebraically degenerate, k_μ being a multiple Debever-Penrose vector.

The Goldberg-Sachs [3] theorem immediately implies that the null field k^μ is not only geodesic but also shear-free.

We now introduce at each point of space-time a quasiorthogonal tetrad of null vectors e_a^μ , where Latin or tetrad suffixes range and sum over 1, 2, 3, 4, and serve to label the different vectors of the tetrad, while Greek suffixes are, as before, tensor suffixes. The tetrad vectors e_1^μ and e_2^μ are complex conjugates,

$$(2.12) \quad e_2^\mu = \bar{e}_1^\mu ,$$

e_3^μ and

$$(2.13) \quad e_4^\mu = k^\mu$$

are real, and they satisfy the quasiorthogonality relations

$$(2.14) \quad e_a^\mu e_{b\mu} = g_{ab} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = g^{ab} .$$

We define the Ricci rotation coefficients by

$$(2.15) \quad \Gamma_{abc} = -e_{a\mu;\nu} e_b^\mu e_c^\nu = -\Gamma_{bac} .$$

The geodesic and shear-free character of $e_4^\mu = k^\mu$ is expressed analytically by

$$(2.16) \quad \Gamma_{m44} = \Gamma_{411} = \Gamma_{422} = 0.$$

The rotation coefficients

$$(2.17) \quad \Gamma_{241} = z = \theta - i\omega, \quad \Gamma_{142} = \bar{z} = \theta + i\omega$$

are Sachs' complex expansion [cf. Goldberg and Sachs [3], eq. (1.8)], $\theta = \text{Re}[z]$ being the expansion rate of the congruence of null geodesics which have k^μ as tangents, and $\omega = -\text{Im}[z]$ being the rotation rate. Throughout the following we shall consider only the general case

$$(2.18) \quad z \neq 0.$$

The tetrad or *slash* derivative of an invariant is defined by

$$(2.19) \quad T_{/a} = T_{;\mu} e_a^\mu = T_{;\mu} e_a^\mu.$$

By (2.9) and (2.13)

$$(2.20) \quad T_{/4} = \dot{T}.$$

For the commutator of two successive slash derivatives we obtain

$$(2.21) \quad T_{/ab} - T_{/ba} = T_{/m}(\Gamma_{ab}^m - \Gamma_{ba}^m),$$

where

$$(2.22) \quad \Gamma_{ab}^m = g^{mn} \Gamma_{nab}.$$

The tetrad components of the Ricci tensor are

$$(2.23) \quad R_{bc} = R_{\mu\nu} e_b^\mu e_c^\nu = \\ = \Gamma_{ba/c}^a - \Gamma_{bc/a}^a + \Gamma_{ba}^m \Gamma_{mc}^a - \Gamma_{bc}^m \Gamma_{ma}^a + \Gamma_{bm}^a (\Gamma_{ca}^m - \Gamma_{ac}^m).$$

3. - The field equations.

We choose null co-ordinates in Minkowski space, complex conjugate co-ordinates $x^1 = \xi$, $x^2 = \bar{\xi}$, and real co-ordinates $x^3 = v$, $x^4 = u$, such

that

$$(3.1) \quad \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and

$$(3.2) \quad ds^2 = g_{\mu\nu} dx^\mu dx^\nu = 2 d\zeta d\bar{\zeta} + 2 du dv + 2H(k_\mu dx^\mu)^2.$$

A general field or real null vectors in Minkowski space [cf. eq. (1.6)] is given by (*)

$$(3.3) \quad k_\mu dx^\mu = e_{4\mu} dx^\mu = du + Y d\bar{\zeta} + \bar{Y} d\zeta - Y \bar{Y} dv,$$

where Y is an arbitrary complex function of position. The tetrad is now completed as follows,

$$(3.4) \quad \begin{cases} e_{1\mu} dx^\mu = d\bar{\zeta} - \bar{Y} dv, \\ e_{2\mu} dx^\mu = d\zeta - Y dv, \\ e_{3\mu} dx^\mu = dv - H k_\mu dx^\mu, \end{cases}$$

and is easily seen to satisfy the quasiorthogonality relations (2.14).

The conditions (2.16) for k_μ to be geodesic and shear-free become

$$(3.5) \quad Y_{/2} = Y_{/4} = \bar{Y}_{/1} = \bar{Y}_{/4} = 0.$$

We then find that

$$(3.6) \quad \Gamma_{m14} = \Gamma_{m24} = \Gamma_{m34} = 0.$$

Geometrically, this means that the tetrad vectors (3.4) are propagated parallelly along each curve of the congruence of null geodesics which have k^μ as tangents. Sachs' complex expansion, eq. (2.17), becomes

$$(3.7) \quad z = Y_{/1}, \quad \bar{z} = \bar{Y}_{/2}.$$

(*) The special case $k_\mu dx^\mu = dv$ can be included by a limiting process, e.g., by starting with $k_\mu/(1+Y\bar{Y})$ and then letting $Y \rightarrow \infty$. The complex function Y is the ratio of the two components of the spinor which corresponds to the null vector k_μ .

From here on, numerical suffixes on the Ricci tensor will always refer to its tetrad components R_{bc} as given by eq. (2.23).

The geodesic and shear-free condition (3.5) ensures that the following five field equations are satisfied:

$$(3.8) \quad R_{11} = R_{22} = R_{44} = R_{14} = R_{24} = 0 .$$

The field equation $R_{12} = 0$ gives

$$(3.9) \quad H = e^{3P}(z + \bar{z}) , \quad P_{;4} = 0 ,$$

where P , like H , is real. The field equation $R_{34} = 0$ is now automatically satisfied. The field equation $R_{31} = 0$ and its complex conjugate $R_{32} = 0$ give

$$(3.10) \quad P_{;1} = (z/\bar{z}) \bar{Y}_{;3} , \quad P_{;2} = (\bar{z}/z) Y_{;3} .$$

Finally, the field equation $R_{33} = 0$ gives

$$(3.11) \quad P_{;3} = \left(\frac{1}{z} + \frac{1}{\bar{z}} \right) Y_{;3} \bar{Y}_{;3} .$$

It is well worth pointing out that the calculation giving these results are by no means simple. In each of the above equations, all previously mentioned field equations have been assumed to hold, and frequent use has been made of the commutator relation (2.21) for slash derivatives.

It is now easy to show that P , Y and $\bar{Y}$ satisfy the pair of partial differential equations

$$(3.12) \quad X_{;4} = 0 , \quad X_{;3} - X_{;1} Y_{;3}/z - X_{;2} \bar{Y}_{;3}/\bar{z} = 0 ,$$

and that Y and $\bar{Y}$ are functionally independent because $z \neq 0$ has been assumed. It follows that P is a function of Y and $\bar{Y}$:

$$(3.13) \quad P = P(Y, \bar{Y}) ,$$

and that the field eqs. (3.9), (3.10), (3.11) simplify to

$$(3.14) \quad P_Y = \bar{Y}_{;3}/\bar{z} , \quad P_{\bar{Y}} = Y_{;3}/z .$$

Differentiating the first of these equations in the direction of the tetrad vector e_1^μ , we obtain, again using commutator relations for the

slash derivatives, $zP_{\bar{Y}Y} = zP_Y^2$, or equivalently

$$(3.15) \quad (\exp[-P])_{\bar{Y}Y} = 0.$$

Similarly $(\exp[-P])_{\bar{Y}\bar{Y}} = 0$, so that $\exp[-P]$ must be bilinear in Y and $\bar{Y}$:

$$(3.16) \quad \exp[-P] = b + \bar{a}\bar{Y} + aY + cY\bar{Y},$$

b and c being arbitrary real constants, and a and $\bar{a}$ arbitrary complex conjugate constants.

It is now easy to show that the operator

$$(3.17) \quad K = K^\mu \frac{\partial}{\partial x^\mu} = b \frac{\partial}{\partial u} + a \frac{\partial}{\partial \bar{\xi}} + \bar{a} \frac{\partial}{\partial \xi} + c \frac{\partial}{\partial v}$$

satisfies

$$(3.18) \quad KY = 0, \quad K\bar{Y} = 0, \quad Kz = KY_{,1} = 0, \quad K\bar{z} = K\bar{Y}_{,2} = 0.$$

Thus, by eqs. (3.2), (3.3), (3.9) and (3.13), K^μ is a Killing vector of our curved space-time with metric $g_{\mu\nu}$. Also, since K^μ has constant components, it is at the same time a translational Killing vector of the flat Minkowski space with metric $\eta_{\mu\nu}$.

4. - The general solution.

We are still free to perform Lorentz transformations which preserve the form $ds_0^2 = 2d\xi d\bar{\xi} + 2du dv$ of the Minkowski metric. By a suitable choice of such a Lorentz transformation, we can simplify our solutions as follows:

$$\text{Case I:} \quad \eta_{\mu\nu} K^\mu K^\nu < 0, \quad a = \bar{a} = 0, \quad b = c > 0.$$

Then $K^\mu = b(0, 0, -1, 1)$ points along the time axis of Minkowski space. Equations (3.5) and (3.18) give

$$(4.1) \quad \frac{\partial Y}{\partial v} + Y \frac{\partial Y}{\partial \bar{\xi}} = 0, \quad \frac{\partial Y}{\partial \bar{\xi}} - Y \frac{\partial Y}{\partial u} = 0, \quad \frac{\partial Y}{\partial u} - \frac{\partial Y}{\partial v} = 0.$$

The general solution is

$$(4.2) \quad F = 0,$$

$$(4.3) \quad F(Y, \bar{\xi}, \xi, u + v) = \Phi(Y) + [Y^2 \bar{\xi} - \xi + (u + v)Y],$$

where Φ is an arbitrary analytic function of the complex variable Y .

It is now easy to calculate

$$(4.4) \quad z = Y_{,1} = -(1 + Y\bar{Y})/F_Y,$$

$$(4.5) \quad \exp[3P] = \sqrt{2}m(1 + Y\bar{Y})^{-3},$$

where F_Y is the partial derivative with respect to Y of the function (4.3), where $m = 2^{-\frac{1}{2}}b^{-3}$ is an arbitrary real constant, and where Y is determined as a function of the co-ordinates by the equation $F = 0$. The solution takes the final form

$$(4.6) \quad ds^2 = 2 d\zeta d\bar{\zeta} + 2 du dv - 4\sqrt{2}m \operatorname{Re} \left[\frac{1}{F_Y} \right] \cdot \left[\frac{du + Y d\bar{\zeta} + \bar{Y} d\zeta - Y\bar{Y} dv}{1 + Y\bar{Y}} \right]^2.$$

$$\text{Case II:} \quad \eta_{\mu\nu} K^\mu K^\nu > 0, \quad a = \bar{a} = 0, \quad b = -c > 0.$$

Then $K_\mu = b(0, 0, 1, 1)$ points along a space axis of Minkowski space.

Arguments similar to those above, give the solution in final form

$$(4.7) \quad ds^2 = 2 d\zeta d\bar{\zeta} + 2 du dv + 4\sqrt{2}m \operatorname{Re} \left[\frac{1}{F_Y} \right] \cdot \left[\frac{du + Y d\bar{\zeta} + \bar{Y} d\zeta - Y\bar{Y} dv}{1 - Y\bar{Y}} \right]^2,$$

where $m = 2^{-\frac{1}{2}}b^{-3}$ is an arbitrary real constant, Y is a complex variable, F_Y is the partial derivative with respect to Y of the function

$$(4.8) \quad F(Y, \bar{\zeta}, \zeta, u - v) = \Phi(Y) - [Y^2\bar{\zeta} + \zeta + (u - v)Y],$$

where Φ is an arbitrary analytic function of Y , and where Y is determined as a function of the co-ordinates by the equation

$$(4.9) \quad F = 0, \quad \Phi(Y) = Y^2\bar{\zeta} + \zeta + (u - v)Y.$$

$$\text{Case III:} \quad \eta_{\mu\nu} K^\mu K^\nu = 0, \quad a = \bar{a} = c = 0, \quad b > 0, \quad K^\mu = b(0, 0, 0, 1).$$

With the same notation and explanations as above, we have the solution in final form

$$(4.10) \quad ds^2 = 2 d\zeta d\bar{\zeta} + 2 du dv - 4\sqrt{2}m \operatorname{Re} \left[\frac{1}{F_Y} \right] \cdot (du + Y d\bar{\zeta} + \bar{Y} d\zeta - Y\bar{Y} dv)^2,$$

$$(4.11) \quad F(Y, \bar{\zeta}, \zeta, v) = \Phi(Y) + \zeta - vY,$$

$$(4.12) \quad F = 0, \quad \Phi(Y) = vY - \zeta.$$

5. - Gravitational field of rotating body and Schwarzschild solutions.

We consider here a particular solution with a Killing vector which is time-like with respect to the Minkowski metric $\eta_{\mu\nu}$. Equation (4.2) can be solved easily and explicitly if Φ is a quadratic polynomial in Y . We have chosen the direction of the time axis of Minkowski space so as to simplify the general solution to the form (4.6). We are, however, still free to perform translations and rotations in the 3-space orthogonal to the time direction. By a suitable choice of such a translation and rotation, the function F , eq. (4.3), with an arbitrary quadratic polynomial $\Phi(Y)$ can be reduced to the standard form (*)

$$(5.1) \quad \Phi(Y) = -\sqrt{2}iaY, \quad a \geq 0.$$

We introduce rectangular Cartesian co-ordinates x, y, z, t in Minkowski space which are related to the null co-ordinates as follows

$$(5.2) \quad \sqrt{2}\zeta = x + iy, \quad \sqrt{2}v = z - t, \quad \sqrt{2}u = z + t.$$

Then the function F of eq. (4.3) is given by

$$(5.3) \quad F = 2^{-4}[(x - iy)Y^2 + 2(z - ia)Y - (x + iy)].$$

We define a real function ϱ by

$$(5.4) \quad \frac{x^2 + y^2}{\varrho^2 + a^2} + \frac{z^2}{\varrho^2} = 1.$$

The solution of $F = 0$ is

$$(5.5) \quad Y = \frac{(\varrho - z)(\varrho + ia)}{\varrho(x - iy)},$$

and for this Y we have

$$(5.6) \quad F_Y = \sqrt{2}[(x - iy)Y + (z - ia)] = \sqrt{2}(\varrho^2 - iaz)/\varrho,$$

$$(5.7) \quad Y\bar{Y} = \frac{\varrho - z}{\varrho + z}.$$

(*) In this section we have changed notation slightly: a is now a real number which has nothing to do with the complex a of eq. (3.16); z is now one of the rectangular Cartesian co-ordinates x, y, z, t in Minkowski space, and not Sachs' complex expansion, eq. (2.17).

Substituting into eq. (4.6), we obtain

$$(5.8) \quad ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \\ + \frac{2m\varrho^3}{\varrho^4 + a^2 z^2} \left[dt + \frac{z}{\varrho} dz + \frac{\varrho}{\varrho^2 + a^2} (x dx + y dy) + \frac{a}{\varrho^2 + a^2} (x dy - y dx) \right]^2.$$

This is the exterior gravitational field of a rotating body [1]. When we put $a=0$, so that, by eq. (5.4), $\varrho = r = (x^2 + y^2 + z^2)^{\frac{1}{2}}$, this reduces to the Schwarzschild metric

$$(5.9) \quad ds^2 = dx^2 + dy^2 + dz^2 - dt^2 + \frac{2m}{r} (dr + dt)^2$$

in a form first given by Eddington [4].

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INTERVENTI E DISCUSSIONI

— W. D. BEIGLBÖCK:

You gave us two solutions in your class, that have a fairly simple physical interpretation—namely the Schwarzschild and the Kerr solutions. Do you know whether there are any further solutions that would allow a physical interpretation?

— A. SCHILD:

The answer is «no». When $\Phi(Y)$ is not a quadratic polynomial the solutions are complicated, and we have not examined them.

— A. KOMAR:

Ehlers, in his Royaumont paper, presented a general procedure for obtaining a Ricci flat stationary space from a given Ricci flat static space. What is the relationship (if any) between his procedure and yours?

— J. EHLERS:

The Schild-Kerr procedure is a definite generalization. The earlier procedure gives only one parametric sub-family of the Kerr-metrics.

— A. SCHILD:

Answer by Ehlers.

— J. EHLERS:

If my method of constructing stationary vacuum fields from static ones (described in my Royaumont lecture) is applied to the Schwarzschild solution a one-parameter (sub-) family of the Kerr-solution is obtained.

2) In connection with the problem of filling matter into the Kerr-solution I should like to point out: If the matter is to be a perfect fluid,

$$T_{\mu\nu} = (\varrho + p)u_\mu u_\nu - pg_{\mu\nu}, \quad (u_\lambda u^\lambda = 1),$$

and if the interior metric is to be stationary (like the exterior one) with Killing vector

$$\xi^\lambda = e^\sigma u^\lambda$$

and if, moreover, g_{ab} is supposed to satisfy the Lichnerowicz junction conditions along the (hyper-) surface S of the body with ξ^λ of class C^1 , then S is given by $U = \text{const}$. In fact, it follows from the above assumption that

$$U_{;\lambda\mu} u^\mu = -U_{,\lambda}, \quad p_{,\lambda} u^\lambda = 0,$$

consequently because of « Euler's » equation

$$(\varrho + p)u_{\lambda;\mu} u^\mu = (\delta_\lambda^\mu - u^\mu u_\lambda)p_{,\mu}$$

we obtain $(\varrho + p)du = -dp$ which implies, since $p = 0$ along S , that $u = \text{const}$ along S also. In consequence of this fact it is possible to study the « shape » of the field-producing body by means of data of the exterior field alone, at least under the hydrostatic circumstances stated above. In this way one may be able to get further support of the assumption that this exterior solution actually corresponds to a body in a state of stationary rotation.

— A. SCHILD:

This is a very interesting remark. We shall look at the surfaces on which the stationary Killing vector has constant magnitude. Since the Kerr metric is known explicitly, this should be a straight forward problem.

[Alla discussione parteciparono anche G. MC VITTIE, E. T. NEWMAN, C. MØLLER, H. J. TREDER, R. DEBEVER.]

— A. SCHILD (*additional remark*):

Professor Misner in a letter, and Professors Debever and Lichnerowicz at this meeting, kindly drew my attention to papers by Vaidya (*Nature* (1953)) and J. Hély (*Compt. Rend.* (1960)) which examine some space-times with electromagnetic radiation where the metric has the form of our eq. (1.1).

Radiation and Observables. (*)

P. G. BERGMANN

Yeshiva University - New York, and Syracuse University - Syracuse

The general topic of our meeting is « General Relativity: Problems of Energy and Gravitational Waves ». Within this framework I have chosen to discuss with you the concept of observables as it has developed in general relativity, and more particularly what we may learn about it from the consideration of radiative solutions of the field equations. I am afraid that much of what I have to say may be known to many of you; but, as a result, we all may profit more from the discussion that is to follow each of the talks.

Let me begin by recalling to you the evolution of the concept of observables that emerged in general relativity about ten years ago, and in connection with the program of quantization. In the course of our attempts to quantize the gravitational field in rough analogy to the Poisson-bracket formalism that is possible in the c -number (non-quantum) version of the theory, we gradually realized that the constraints inherent in the Hamiltonian formulation of Einstein's field equations would present major obstacles. True, a similar situation prevails in quantum electrodynamics; but the very nearly linear character of that theory makes it possible to separate the « gauge variables » from the intrinsic degrees of freedom without excessive manipulatory efforts; a corresponding procedure in general relativity is extremely awkward, to say the least.

Gradually we realized that physically it does not make sense to construct Hermitian Hilbert space operators corresponding to every physical variable in the theory. A quantum observable possesses expectation values, and an eigenvalue spectrum, discrete or continuous. Many field variables in a theory such as general relativity are not possibly observable, because their numerical values are devoid of any physical meaning. By performing arbitrary co-ordinate transformations we may give almost

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any value we want to the component of a tensor, for instance, or even to a scalar, at a world point identified by the values of its four space-time co-ordinates, without thereby providing any substantive information about the physical situation in which this value occurs. Because of the freedom of curvilinear co-ordinate transformations we eventually came to the conclusion that the only quantities that ought to possess expectation values are invariants, that is to say quantities that have the same values regardless of the choice of (conventional) co-ordinate system. Because of a frequent confusion in terminology let me say, parenthetically, that I shall distinguish between invariants and scalars. A scalar is a one-component field defined on the four-dimensional space-time manifold which obeys the infinitesimal Lie-transformation law

$$(1) \qquad \delta^* U = - U_{,e} \delta x^e$$

whereas an invariant obeys the law $\delta^* U = 0$.

Because of the special nature of the curvilinear co-ordinate transformations in general relativity, which include the possibility of a transformation of the time co-ordinate, an invariant is necessarily a constant of the motion; and thus we concluded that an observable in general relativity is *ipso facto* a constant of the motion. In a Hamiltonian formalism all properties of the gravitational field can be expressed in terms of appropriate Cauchy data on a space-like three-dimensional hypersurface; Dirac has shown that the twelve field variables g_{mn}, p^{mn} are suitable as Cauchy data provided they satisfy at each point of the hypersurface the four constraint conditions $H_s = 0, H_L = 0$. Obviously, the value of any constant of the motion is contained in principle in these Cauchy data, though the data are not by themselves invariant: aside from intrinsic data they contain information about the location of the initial-value hypersurface and about the choice of three-dimensional co-ordinates on that hypersurface. But we may conclude that any observable, or constant of the motion, may be represented in principle as a functional on a three-dimensional hypersurface. An observable may then be characterized as a functional that has vanishing Poisson brackets with all of Dirac's constraints; this is because the constraints are the generators of the infinitesimal co-ordinate transformations. In Dirac's terminology an observable is a first-class quantity. The first-class variables form a Lie algebra among themselves, which includes the constraints; hence there is no obvious formal impediment to a program of representing this Lie algebra in terms of Hermitian Hilbert-space operators. Needless to say, the realization of this program in practical terms is another matter.

This is not the occasion to discuss the various techniques that are being tried to construct actual observables along the lines that I have

just sketched. Rather, I should like to discuss the changes in the concept of observables that suggest themselves in the restricted program that focuses attention on a special class of solutions of general relativity that we might call the radiative solutions. I am referring to the techniques that were initiated by H. Bondi and his coworkers, which have been elaborated and further clarified, among others, by E. T. Newman and his coworkers, by R. Penrose, and by R. K. Sachs. We are dealing here with solutions of the field equations that are restricted by boundary conditions imposed at infinity, and more precisely at infinity in all light-like directions. The solutions considered are not generally « global » in that we admit arbitrary sources at all times in a finite region surrounding the origin of the spatial co-ordinates. In other words, we are looking for solutions in the neighborhood of light-like infinity which at infinity are subject to certain restrictions; these restrictions might be described as « asymptotic flatness » provided we remember that this is rather vague language and that for any precise analysis we require a much more specific statement.

Sachs has investigated the transformation group of co-ordinates that preserve the asymptotic conditions imposed on the radiative fields, and he has found that this group involves six parameters, as they might occur in a Lie group, and one arbitrary function of two variables. He has called this group the generalized Bondi-Metzner group, or GBM group. I shall follow his terminology.

Intuitively we may describe the GBM group as being composed of homogeneous Lorentz transformations and of the « supertranslations » (this term also was originated by Sachs). If we describe the four-dimensional manifold by the four co-ordinates u (retarded time), r (luminosity distance), θ and φ (angles on the celestial sphere), then light-like infinity is characterized by r going to infinity, with the remaining three co-ordinates retaining their finite values. Thus light-like infinity is a three-dimensional manifold, representable as the product of a spherical surface (two-dimensional surface of the three-dimensional sphere) and the one-dimensional time axis; u, θ, φ are suitable co-ordinates; or we may prefer to perform a conformal mapping of the spherical surface on the complex plane and to replace θ and φ by the complex variable z .

Now we can describe the infinitesimal GBM group in terms of the infinitesimal transformations of the co-ordinates u and z , where u is real and z complex. This infinitesimal transformation law is the following:

$$(2) \quad \delta z = \alpha + \beta z + \gamma z^2, \quad \delta u = f(z + \bar{z}, z\bar{z}).$$

α, β, γ are three arbitrary complex parameters, which describe infinitesimal (real) Lorentz transformations, whereas $f(\dots)$ is an arbitrary real

function of its two real arguments. It is a routine matter to verify that the commutator of two infinitesimal transformations of the type (2) is a transformation of the same type. Actually Sachs has done much more than just look at the group germ. And I shall not reproduce his beautiful work here.

I now come to the principal topic of my talk. What is the appropriate definition of an observable in the presence of an invariance group of the type (2)? Invariance groups of such a mixed character occur elsewhere in physics, and we might attempt to obtain some guidance from these situations, at least one of which is familiar to all of us; that is the invariance group of electrodynamics. Lorentz-Maxwell theory is invariant both under Lorentz transformations and under gauge transformations. Whereas the Lorentz transformations by themselves form a Lie group, the gauge transformations (of the second kind, of course) involve an arbitrary function of the four co-ordinates. The commutator of a Lorentz transformation and a gauge transformation is a gauge transformation, just as in the GBM group the commutator of a Lorentz transformation and a supertranslation is a supertranslation. How do we characterize the observables of electrodynamics?

Intuitively, and guided by all our experience of many years, we should say that an observable must be gauge-invariant, but that it need not be Lorentz-invariant. Indeed, we generally assume that we can measure the value of a component of the electric field strength at a specified world point, but that there is no meaning attached, say, to the value of a component of the vector potential at the same world point. Can we adduce arguments of a formal nature that will support this intuitive judgement?

There is, first of all, the Cauchy problem. Given an adequate set of Cauchy data, the Lorentz-Maxwell equations enable us to predict, by integration, the values of all gauge-invariant quantities off the initial hypersurface. Lorentz invariance is not necessary. Moreover, these predictable quantities need not be constants of the motion. Because they are obtainable from Cauchy data, and because the initial-value hypersurface may be chosen to correspond to any particular value of the time co-ordinate, t_0 , the value of a gauge-invariant field variable at a chosen world point x may be expressed as a functional on a hypersurface $t = t_0$ for all values of t_0 . Though the *form* of the functional will depend on the choice of t_0 , its *value* will not; and in this sense the functional *is* a constant of the motion. We cannot perform a similar construction for a variable that is not gauge-invariant, because the field equations of the electromagnetic field will not relate the value of such a variable at one time to any set of Cauchy data at another time.

Let us be a bit specific about our choice of Cauchy data. Let us choose

the initial-value hypersurface so that it corresponds to very definite values of the co-ordinates. For instance, we might specify that the hypersurface be the hyperplane $t = 0$. On this hyperplane we shall choose a set of initial values, either for the potential or for the field strengths. If we wish to conform to a standard Hamiltonian formulation of the Cauchy problem, we may choose, for instance, the values of the vector potential at the time $t = 0$ at all space points, and also the values of the canonically conjugate momentum density; these equal, of course, the negative electric field strength. These Cauchy data are subject to one constraint at each (space) point,

$$(3) \qquad \operatorname{div} \pi = 0 ,$$

if we treat the vacuum case, *i.e.* the case in which electric charges are absent. With such a set of permissible Cauchy data, our problem is well-set; values of all gauge-invariant quantities are determined throughout the four-dimensional Minkowski manifold; purposely, I have not chosen Cauchy data that are gauge-invariant. There are, of course, gauge transformations that change the numerical values of the A -field without affecting the physical situation.

If I apply some permissible transformation to my whole situation, I immediately discover a major difference between Lorentz transformations and gauge transformations: every conceivable Lorentz transformation will change the values of my Cauchy data; but there are gauge transformations that will leave them unchanged. The latter are those gauge transformations in which the scalar whose derivatives are added to the potentials vanishes on the initial-value hyperplane, along with its first time derivative. In view of the fact that this scalar field need not be analytic, the choice of the scalar on any domain off the initial-value hyperplane remains arbitrary even in the presence of restrictions imposed on it on the hyperplane. Accordingly, I may change the value of any off-hyperplane variable that is not gauge-invariant by a gauge transformation that leaves the Cauchy data unchanged. I cannot perform any such feat with Lorentz transformations.

From the foregoing, it appears that the chief difference between Lorentz transformations and gauge transformations is that the latter involve arbitrary functions of the time co-ordinate, whereas the former do not. It is not decisive that one set of transformations is described by a finite set of parameters (ten in the case of the inhomogeneous Lorentz group), the other by functions. To test this distinction we shall look at a case whose invariance group involves arbitrary functions which do not involve the time co-ordinate as an argument. Let us consider again electrodynamics, but electrodynamics whose gauge frame has been restricted

by the gauge condition that the scalar potential is to vanish, $\varphi = 0$. (We shall not adopt the so-called radiation gauge, in which the divergence of the $\mathbf{A}$ -field vanishes as well). With the minimal gauge restriction imposed, we retain a group of gauge transformations in which the scalar field describing the gauge transformations is an arbitrary function of the three spatial co-ordinates $x^1, \dots, x^3$, but independent of t .

If we write the field equations in terms of the canonical variables, they will be:

$$(4) \quad \begin{cases} \frac{\partial \pi}{\partial t} = -\text{curl curl } \mathbf{A}, \\ \frac{\partial \mathbf{A}}{\partial t} = \boldsymbol{\pi}. \end{cases}$$

There will remain one constraint per space point, which is unchanged from the previous case:

$$(5) \quad \text{div } \boldsymbol{\pi} = 0.$$

If we perform a permissible gauge transformation, the values of $\mathbf{A}$ will change in that it will add the gradient of a time-independent but otherwise arbitrary scalar field. This addition will have no effect on the right-hand side of the first eq. (4), nor will it affect the left-hand side of the second eq. (4). The constraint (5) will be preserved. But it is clear that any gradient field added to the $\mathbf{A}$ -field on the hyperplane $t=0$ will necessarily appear as an addendum to the vector potential at all other times t , and vice versa. Hence the Cauchy data and the field elsewhere determine each other completely, in full agreement with the invariance group that we have postulated.

To summarize our argument: observables must be invariant with respect to all transformations that involve arbitrary functions of the time but need not be invariant with respect to other transformations belonging to the invariance group of the theory.

Perhaps I should make two other comments that illuminate this result. The first refers to the structure of such mixed invariance groups. Whenever we are confronted by a group of transformations some of whose members involve arbitrary functions of the time whereas others do not, the commutator between two infinitesimal transformations one of which involves such an arbitrary function *as a factor* in the transformation law will either vanish or be itself one of the transformations having an arbitrary function of the time as a factor. Hence those members of the original transformation group that have an arbitrary function of the time as a factor (in the *infinitesimal* transformation law) will form a normal subgroup. The factor group will be free of such

arbitrary functions of the time. What we require of our observables is that they be invariant with respect to the normal subgroup; we do not require them to be invariant with respect to the factor group. In general, then, the transformation law of an observable is a representation of the invariance group, but it can be faithful, at best, only to the factor group. In the Lorentz-plus-gauge group the gauge group represents the normal subgroup, and the Lorentz group the factor group.

The other comment is concerned with the generators of the invariance group. Whenever we are confronted with an invariance group that is continuous, then Noether's theorems tell us that its generators are constants of the motion, or that the generating field obeys a conservation law (equation of continuity). A generator that is a constant of the motion but which also contains an arbitrary function of the time as a factor must vanish; this is the standard argument leading to the conclusion that the generators of invariance transformations involving arbitrary functions of the time are constraints. Generators of other invariance transformations need not be constraints, but they must commute with the constraints, *i.e.* they must be first-class quantities (in Dirac's terminology) as well as constants of the motion. Again we may recover the previous result, but now in terms of the technology of the canonical formalism. If some of the generators of the invariance group are constraints, whereas others are nonvanishing constants of the motion, then all of them form a Lie algebra. The constraints form a Lie sub-algebra, and the remainder form a Lie algebra which is the equivalent of a factor group: the commutator between any of the nonvanishing generators (defined up to a linear combination of added constraints) is another generator belonging to the Lie algebra which also is defined up to a linear combination of the constraints. An observable is required to commute with the constraints, but not to commute with the remaining generators of the invariance group. Hence we may characterize observables as generators that map a given physical situation into another situation which is physically feasible (*i.e.* its field variables satisfy all the constraints of the theory); the constants of the motion which generate the remaining invariance transformation may change their values under the canonical mapping. If we were to generate a canonical transformation with the help of a nonobservable generator, such a mapping would destroy the constraints; it would map a physical on a nonphysical situation.

Let me now return to our point of departure, the GBM group. We found that group to be a mixed group, its element being described in part by a set of six parameters, in part by an arbitrary function of two arguments, φ and θ . From our previous discussion it is obvious that the supertranslations form a normal subgroup and that the homogeneous Lorentz group represents the factor group. Through Sachs' work it is

also well known that there is no homomorphism between the GBM group and the inhomogeneous Lorentz group. What should we require of the observables in a theory that is invariant under the GBM group? Formally it is possible to construct quantities that are invariant under the supertranslations but which are Lorentz-covariant; such quantities would be constants of the motion, in that the supertranslations include ordinary translations. But I believe that to restrict the notion of observables to such quantities is not justifiable: the supertranslations depend on an arbitrary function, but the time is not an argument of that function. Once we fix our «supertranslation frame» for one value of the retarded time coordinate u throughout the spherical surface spanned by the complex co-ordinate z , we have exhausted the freedom of choice, and our co-ordinate system is completely fixed. Hence I believe that from the point of view of the general theory which I have presented, any set of variables that form a geometric object under the GBM group should be considered observable even though noninvariant under supertranslations.

Let me stress that my discussion is not concerned with just a formal definition, that is to say which class of quantities we should *call* observables, but with a question that in my opinion possesses considerable physical interest: which quantities should we expect to play the role of Hilbert space operators in the quantum theory of the gravitational field? The answer that I have given here may eventually turn out to require modification, but I believe that the question that I have posed is one in physics, not in semantics.

An immediate application of my proposal is that under it Bondi's news functions will be observables. Being functions of u , as well as of z , the news functions are not invariant under supertranslations, though properly defined they are scalars. According to my definition they will qualify as observables. Both Sachs and Komar have furnished us with commutation relations between the news functions at distinct points. I believe that we now have at hand the tools with which to explore some of the properties of at least a large and interesting class of quantized gravitational fields.

INTERVENTI E DISCUSSIONI

2

— S. BONAZZOLA:

Est-ce que les quantités que M. Bergmann dit observables coïncident ou non avec les grandeurs *standard* introduites par M. Cattaneo et qui sont invariantes

vis à vis du group

$$\begin{aligned} x^\alpha &= x^\alpha(x^\beta) \\ x^4 &= x^4(x^i) \end{aligned} \quad \left(\begin{array}{l} \alpha, \beta = 1, 2, 3 \\ i = 1, 2, 3, 4 \end{array} \right)$$

qui laisse invarié la repère physique?

Moi même j'ai pu démontrer la mesurabilité de ces grandeurs.

— P. G. BERGMANN:

The transformation mentioned by you leaves unchanged the time-like world curves $x^\alpha = \text{constant}$, but not the time scale along such a curve. This is a different transformation invariance from the one envisaged by Dirac (« *D*-invariance »). Quantities that are invariant with respect to this group introduced by Prof. Cattaneo are presumably constants of the motion with respect to that group (because changing the time scale is to have no effect on them), but I do not believe that they are observables in my sense. To have that property they should have to be invariant with respect to unrestricted four-dimensional coordinate transformations. As I have not worked with this particular transformation group, I cannot be certain that my reply is entirely correct.

— W. B. BONNOR:

I wish to make two remarks about Professor Bergmann's discussion of BVM's work. First, I want to point out that it refers only to outgoing radiation. Secondly, I am doubtful whether we should yet speak of radiative *solutions* of Bondi's metric. BVM obtain a great deal of interesting informations from the metric but, as far as I know, no interesting radiative solutions—even of an approximate nature—have been obtained.

— A. KOMAR:

Given the incoming « news functions » at past null infinity (and perhaps additional information at time-like past infinity concerning matter) one would expect that if the « news functions » are to be observable, the outgoing « news function » should be fully determined. However, since the supertranslations at past null infinity cannot be coordinated with the supertranslations at future null infinity, it seems clear that the outgoing « news function » cannot be completely determined by the incoming information. For this reason I question whether the « news function » is in itself an observable.

— P. G. BERGMANN:

In any solution with non-zero mass, changes in the choice of coordinate conditions (*e.g.* retarded versus advanced time) will involve terms that diverge at infinity as $\log r$. The appearance of such logarithmic terms, either in a function or in its arguments, will, of course, prevent a power series expansion in r^{-1} , without indicating necessarily that the solution is physically unacceptable. I believe that further investigation will enable us to control such terms and to render them innocuous.

Mr. Mandred Leiser at Syracuse University, partly under the guidance of Prof. R. Sachs at Texas, is looking into that question right now. Until this

investigation of logarithmic divergences is completed, we can, of course, not promise that it will relieve us of all these difficulties. Incidentally, Prof. Newman has replaced analytic expansions generally by $O(r^{-n})$ estimates, thereby probably obviating many problems of technological nature. Because the news function is not sufficient to characterize the Ricci-flat situation completely (*e.g.* the Schwarzschild solution has a vanishing news function), we should not expect that the incoming news function will determine the outgoing news function by itself. I believe that it will do so if supplemented by additional data such as a mass aspect at a certain time.

— J. A. WHEELER:

One cannot disagree when Professor Bergmann stresses that we want to use all possible methods and clues to understand the content of quantum geometrodynamics. It is consistent with this catholic outlook to ask how the quantum theory of gravitation looks when one formulates it in a form *free of all reference to coordinates*. It has not always been clear that general relativity can be stated in a form free of all reference to coordinates. Neither was it clear for many years that Dirac's theory of the electron is independent of all special choices in the representation of the Dirac gamma matrices. One used to trouble about a conceivable difference in calculated cross-sections according as one used for Γ^a the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

or the matrix

$$\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

or some other matrix. Today we know how to do calculations—and get concrete definite results—by abstract operator analysis, free of all such petty issues of the choice of representation. We also know that in principle all of relativity theory is independent of one's choice of coordinates—and therefore requires no coordinates at all. We even have learned to do a few of the calculations of relativity theory in a way free of all use of coordinates, as for example when we fit together the Friedmann geometry in an interior region and the Schwarzschild geometry of the exterior to give a description of gravitational collapse (work of Bekeedorff and Misner reported in HARRISON, THORNE, WAKANO and WHEELER: *Gravitation Theory and Gravitational Collapse*, University of Chicago Press, scheduled for publication February 1965). However great parts of the subject are still as much encumbered by reference to

coordinates—and to discussion of coordinate transformations—as was relativistic quantum mechanics in the old days. Today it still sometimes said: « Yes, one can dispense with coordinates in principle, but for any practical calculation, or for any physical interpretation, one *must* have coordinates ». In a similar way, when abstract methods first came into relativistic quantum theory, one was at first inclined grudgingly to admit that they were all well and good for abstract questions of principle, but for « real calculation », or for « interpretation of the real physics » one must select some well defined set of 4×4 matrices to represent Dirac's gammas. Today one would no longer attempt to uphold such a claim; and similarly the time will surely come when one will think of reference to *any* coordinate system in general relativity as rather old fashioned!

How one can dispense with coordinates in the description of geometry is shown nowhere as beautifully as in the classical *Théorie des surfaces* of Darboux. Of course quantum theory deals not with a unique geometry but with a probability amplitude for this, that or the other geometry; thus

$$\psi = \psi^{(3)}(\mathcal{G})$$

(cf. for example J. A. WHEELER: *Geometrodynamics and the issue of the Final State*, in C. and B. DE WITT, eds., *Relativity, Groups and Topology*, Gordon and Breach, (New York, 1964)). The wave equation of de Witt (cf. Discussion following the lecture of Ivanenko) tells the rate of change of this state functional. The problem of the quantum theory of geometry are (1) to say all that can be said from the de Witt equation about the dependence of ψ upon $^{(3)}\mathcal{G}$ and (2) to draw conclusions about the physics from (1). This program is only in its beginning today. The full tracing out of the character of quantum geometrodynamics may be a long and difficult task as Professor Bergmann has suggested on more than one occasion—or it may not. However, it may illuminate some of the features of this quest to consider a much simpler problem. Dr. H. Leutwyler of the University of Bern has pointed out (in unpublished work, quoted here by his kindness) that relativity in a $(2+1)$ dimensional spacetime (trivial in *classical* theory because the vanishing of the six R_{ik} implies the vanishing of six R_{ijkl} —and therefore flatness) is not without interest in *quantum* theory. The probability amplitude is a functional of the 2-geometry $^{(2)}\mathcal{G}$ of a spacelike surface. The $(2+1)$ dimensional space in which this 2-surface is imbedded is itself *flat*. Consequently the *extrinsic* curvature (second fundamental form) of this surface is determined by the *intrinsic* geometry $^{(2)}\mathcal{G}$ —a determinism which would not all be the case if the $(2+1)$ dimensional space were *arbitrary*. To know only the intrinsic geometry at a point means knowing the product of the two principal radii of curvature, ϱ_1 , and ϱ_2 , at that point, or knowing the Gaussian curvature $1/\varrho_1\varrho_2$, but *not* knowing ϱ_1 and ϱ_2 individually. But ϱ_1 and ϱ_2 individually are also known when—as here—one also has the extrinsic curvature. Therefore one can evaluate the *mean* curvature

$$\kappa = \left(\frac{1}{\varrho_1}\right) + \left(\frac{1}{\varrho_2}\right).$$

Leutwyler has found considerations that suggest it may be reasonable to think of the state of the system as described by a wave functional of the form

$$\psi^{(2)}(G) = N \exp \left[- \left(\frac{1}{L} \right) \iint \kappa \, d(\text{area}) \right].$$

Here N is a normalization factor and L is a fundamental length associated with the quantum of action. Of course the system in question when analyzed at the classical level has essentially no dynamics. Therefore it does not appear unnatural to think of the system as having a *unique* state functional. The integral in the exponent of the suggested functional is to be distinguished sharply from the integral

$$\iint^{(2)} R \, d(\text{area}) = \iint \left(\frac{2}{\varrho_1 \varrho_2} \right) d(\text{area})$$

which appears in the Gauss-Bonnet formula for the Euler characteristic of the topology. This integral is dimensionless, as befits a topological index number. That it is the *wrong* functional for the exponent of ψ follows not only from the physical considerations of Leutwyler, but also from dimensional considerations. Thus, the constant of Planck must enter in the denominator of the exponent. But this constant is dimensional. Therefore the integral in the numerator must have the *same* dimensions. In other words, the integral *cannot* be the dimensionless integral of the Gauss-Bonnet formula. In any discussion of the nature of quantum geometrodynamics the Leutwyler exponent is of interest because it is completely *nonlocal* in character. It is entirely impossible to predict the *mean* curvature at a point on a 2-geometry from a knowledge of the *Gaussian* curvature at that point. Nor can one evaluate κ even from a knowledge of the entire 2-geometry in a limited region of the point in question. (Example: The 2-geometry of a cord of invitation is entirely Euclidean, and remains so when the cord is bent. The Gaussian curvature remains zero as a consequence of the bending, but the mean curvature is quite different from zero). One thus requires more than a *local* knowledge of the 2-geometry; one must know it *globally* if one is to be able to evaluate the Leutwyler state functional. More significant for the present comments than the global character of the state functional is the freedom which one has in evaluating it to use any coordinate he pleases in $^{(2)}\mathcal{G}$ and even the freedom to *dispense with all coordinates whatsoever*. That the same freedom characterizes the full (3+1)-dimensional theory of general relativity follows directly from the invariance of that theory with respect to co-ordinate transformations. One is sometimes accustomed to use in general relativity the words «invariance with respect to gauge transformation» for «invariance with respect to coordinate transformations». The word «gauge» is brought in to remind one of the analogy between general relativity and electrodynamics. In this connection it is useful

to look at the functional for the ground state of the electromagnetic field (J. A. WHEELER: *Geometrodynamics* Accademic Press, (New York, 1962))

$$\psi(\mathbf{B}(x)) = N \exp \left[- \left(\frac{1}{32} \pi^4 \hbar c \right) \int r_{21}^{-2} \mathbf{B}(x_1) \cdot \mathbf{B}(x_2) d^3x_1 d^3x_2 \right].$$

Here neither the vector potential $\mathbf{A}(x)$ nor the « gauge » of the electromagnetic field makes any appearance. Only the quantity that has physical significance—the magnetic field $\mathbf{B}(x)$ itself—governs the value of the state functional. This example and the Leutwyler state functional remind one of the deep underlying freedom of our physics from all questions of gauge and of coordinates, and urge us to follow the spirit of Darboux and Cartan, in saying all that we can say, and all that we ought to say in quantized general relativity, in a *coordinate free manner*—however needful coordinates are today as a temporary couth!

— S. DESER:

The Leutwyler example (in $2+1$ dimensions) may be a little difficult as an analogy, as the field equations have only the flat solution in the absence of sources. On the more general problem of quantizing without choice of frame, this has certainly not even been done in any rigorous way in electrodynamics, where in fact radiation-like gauges play a preferred role. In relativity, unless one chooses a reference frame, the physics has not been fully specified, and it is not clear why one expects, quantum mechanically, the same results for all frames. This does not seem to be the case already in some simple models (R. ARNOWITT: unpublished).

— P. G. BERGMANN:

I agree completely that quantization without reference to co-ordinates would be most desirable. As this is a most difficult program, technically, I believe we should pursue alternative approaches as well, until one approach at least turns out successful. As for Leutwyler's work, I agree with Deser's remarks that it is difficult to see what quantization of a situation amounts to if the corresponding classical situation possesses zero degrees of freedom

— A. MERCIER:

I am at the present time under the impression that looking for observables and attempting a quantization of General Relativity Theory is a bit like « chasing a rabbit which is not in the land scape ». For, as Fock reminded us, the partition of matter determines the metric, conversely the metric determines the partition of matter, therefore the theory of general relativity is « most deterministic » and would, it seems, resist as a whole all attempts of quantization. Of course, it is the profession of physicists to chase rabbits and therefore it is good to try to construct them with the hope that the things thereby found will at a later stage at least show us what is to be endowed with physical reality. In my opinion, a future theory should include General Relativity and Quantization as elements interpretable as such at lower levels.

— P. G. BERGMANN:

It is difficult for me to understand precisely what is meant by Prof. Fock, whom you have quoted. That the literal Mach principle does not apply in general relativity we know, as there are now several global solutions of Einstein's field equations known which represent empty universes but which are not Minkowski (*i.e.* flat). Hence we have no reason to believe that the gravitational field is determined entirely by the distribution of sources, without any degrees of freedom of its own. As for the corresponding situation in electrodynamics, I believe that the Fokker-Tetrode-Wheeler-Feynman investigations have so far failed to establish whether a classical theory just with relativistic action at a distance is internally consistent or not; and I believe that there are indications that the quantization of such a truncated electromagnetic theory fails to correspond to physical reality. For all these reasons I believe we had better chase the «rabbit» until someone proves that there is none there.

The Gravitons and Einstein's Particle Problem.

H. J. TREDER

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Einstein's idea of a pure geometric field physics requires the existence of global solutions of the Einstein empty space equations $R_{\mu\nu}=0$ that satisfy physical meaningful boundary conditions. This global solutions have to be of such structure that the Einstein equations were satisfied everywhere in the space-time without the appearance of singularities [2, 3].

Of course, the existence of global solutions are depending on the suppositions about the topology of the space-time. But the Einstein equations for the metrical field $g_{\mu\nu}$ give necessarily conditions for the topology of the space-time. So, if one like to prove the existence of global solutions with definite physical properties one have to prove what global structure of the space-time is compatible not only with the physical situation but also with the Einstein equations. According to the meaning of Einstein it is not useful to postulate mathematical suppositions additional to the global satisfaction on the Einstein equations and to the physical initial and boundary conditions. Besides, it is not useful to make suppositions about the topology of the space-time or differentiability hypotheses for the metric which don't result from the Einstein equations themselves.

Besides, one must not postulate *a priori* that the signature of the space-time have the Lorentz-Minkowskian value on each world points. As proved in an earlier paper [10] the Einstein equations permit the turn of the Lorentz-Minkowskian signature — for instance — in a definite signature. In the case the places on which the determinant g of the continuous covariant metrical tensor $g_{\mu\nu}$ was vanished there are not singularities of the Einstein equations. If the determinant g is vanished at a subspace (hypersurface, surface, curve, point) the Einstein equations are satisfied in the meaning of a limes on the subspace if the equations were satisfied in the four-dimensional neighbourhood of the subspace.

If g is vanished — for instance — on the hypersurface $x^1 = 0$ so are [10]

$$(1) \quad \lim_{x^1 \rightarrow 0} \left(\frac{g^2 R_{nv}}{g^2} \right) = \lim_{x^1 \rightarrow 0} (R_{nv}) = 0 .$$

if the field equations are fulfilled as well as for $x^1 < 0$ and for $x^1 > 0$.

Furthermore, we don't require that the second and higher derivatives of the metrical tensor are continuous. The properties of the derivatives are resulted by oneself from the choice of the co-ordinates and from the satisfaction of the Einstein equations. The satisfaction of the integral forme [9]:

$$(2) \quad \int_{G_4} R_{nv} d^4x = 0$$

of the Einstein equations for each domain of integration G_4 are required in regard of the first derivatives. — We permit so the existence of discontinuities on nonisotropic hypersurface $z = \text{const}$ only in this form

$$(3a) \quad \left[\frac{\partial}{\partial z} g_{nv} \right] = a_n p_v + a_v p_n \quad \text{with } p_n p^n \neq 0$$

but on isotropic hypersurface $z = \text{const}$ in this form:

$$(3b) \quad \left\{ \begin{array}{l} \left[\frac{\partial}{\partial z} g_{nv} \right] = a_n p_v + a_v p_n + \pi_n \tilde{\pi}_v + \tilde{\pi}_n \pi_v \\ \text{with} \\ \pi_n p^n = \tilde{\pi}_n p^n = \pi_n \tilde{\pi}^n = p_n p^n = 0 . \end{array} \right.$$

The covariant metric g_{nv} itself shall be continuous.

The programme for a geometrical field theory by Einstein have to be treated with these general suppositions.

This programme contents two particular aspects:

- 1) the theory of the free gravitational waves;
- 2) the question for gravitational models of self-consistence elementary particles. (Einstein's particle problem.)

Both problems are in one sense one problem because according to Einstein the relativistic theory of gravitation must contents the quantization of the gravitational field itself. The quanta of the gravitational fields — the gravitons — have to be global solutions of the Einstein equations.

This programme of general relativity have to correspond with the quantum theory of fields. Therefore, the properties of the gravitons which result from the formal quantization of the gravitational field do have to result from the gravitons like solutions in Einstein's particle problem. Therefore, the properties of the gravitational fields which result from the quantization of the exact Einstein equations are clues to the necessary physical and geometrical properties of solutions of the Einstein equations which correspond with free gravitons. De Witt [1], Regge [7], Wheeler [14] and others have proved that the quantization of the exact equations leads to the existence of a smallest length-Planck's elementary length:

$$l = \sqrt{\frac{\hbar f}{c^3}} \approx 10^{-33} \text{ cm} .$$

The causal nexus breaks down in domains of the space-time with measures smaller than Planck's length. And this break down is a consequence of the change of the signature. An argument for this is [12]:

If we try to measure a length smaller than Planck's length l the Lorentz signature of space-time will be destroyed then.

This results from the local strong gravitational fields which are produced by the particles with de Broglie waves lengths smaller than l . The space-time of this domains get a definite signature with $s = -4$.

This peculiarity of the quantum theory of gravitational fields leads to the conjecture that the gravitons are corresponding with space-time domains for which the metric has a definite signature as a result of the strength of the gravitational field [12]. Any local metric with definite signature are corresponding with strong gravitational fields in absolute sense: there are not a physical co-ordinate-system in which such a metric becomes Minkowskian on a point. So far the quantum theory of gravitational fields touches the ideas of quantum physics that in a very small space-time domain the special relativity becomes invalid because the Minkowskian causality-cones can't be definite in this domains. In the quantum theory were discussed models of elementary particles too for which models we have in the exterior domain of a four-dimensional cylinder a Minkowskian metric ($g_{nv} = \eta_{nv}$) but in the interior domain we have a four-dimensional euclidian metric ($g_{nv} = -\delta_{nv}$).

According the remarks about the strength of gravitational fields with definite signature, only the general relativity is able to prove the consistence of such models. It can be proved that the general relativity permits models of this kind, indeed. But the well-known particle models are not gravitons yet for they define particles with restsystems. As particular solutions of Einstein's particle problem these models may be patterns for the treatment of the graviton problem.

These models of selfconsistence particles with restmass stipulate for a stationary or time-periodical metric that we are able to put:

$$(4) \quad g_{nv,0} = 0 \quad \text{or} \quad \overline{g_{nv,0}} = \frac{1}{T} \int_0^T g_{nv,0} dx^0 = 0 \quad (T = \text{period}).$$

On the space-like hypersurface $x^0 = \text{const}$ the boundary conditions

$$(5) \quad R_{nv\lambda}^\tau \rightarrow 0 \left(\frac{1}{r^3} \right) \quad \text{for } r \rightarrow \infty \quad (r^2 = x_i x^i) \\ (x^i = \text{quasi-Cartesian coordinates})$$

have to be satisfied. As a simple case we are willing to discuss a static gravitational field with

$$(6a) \quad g_{nv,0} = 0, \quad g_{i0} = 0, \quad i = 1, 2, 3.$$

Then the time-like Killing-vector is:

$$(6b) \quad \xi^n = \delta_0^n, \quad \xi_n = g_{00} \delta_0^n.$$

This vector is laying on the hypersurface $x^1 = \text{const}$.

We interpret x^1 like a kind of radial co-ordinate and we postulate that the space-time may have the Lorentz signature for each world domains in the exterior of a cylindrical hypersurface Σ (with $x^1 = 0$). Then the Killing-vector is time-like indeed. Besides, the three-dimensional matrix g_{ik} may be negative definite. We have so:

$$(7) \quad g = |g_{ik}| g_{00} \equiv -\gamma^2 g_{00} < 0, \quad g_{00} = V^2 > 0, \quad \text{for } x^1 > 0.$$

But on the hypersurface Σ the metric g_{nv} may be degenerate and the Killing-vector becomes isotropic. We have:

$$(8a) \quad g_{00} = 0 \quad \text{for } x^1 = 0$$

and therefore on

$$(8b) \quad g = 0, \quad g^{00} = \lim_{x^1 \rightarrow 0} \left(\frac{1}{g_{00}} \right) \rightarrow \infty.$$

The Einstein equations for a static field are:

$$(9) \quad \begin{cases} R_{00} - \frac{V}{\gamma} (g^{ik} \gamma V_{,i})_{,k} = 0, & R_{i0} = 0, \\ R_{kl} = P_{kl} + \frac{1}{V} (V_{,kl} - \Gamma_{kl}^i V_{,i}) = 0 \\ (P_{kl} = \text{Ricci-tensor of the } V_g \quad x^0 = 0). \end{cases}$$

In an earlier paper [10] it was proved that (8) and (9) give for g_{00} the power series in x^1 :

$$(10) \quad g_{00} = \alpha_2(x^2, x^3)(x^1)^2 + \alpha_4(x^2, x^3)(x^1)^4 + \dots$$

(with $g_{12} = g_{13} = 0$ and $g_{11,1} \rightarrow 0$ for $x^1 \rightarrow 0$).

Therefore, the determinant g vanishes on Σ in the second order and g_{00} has a pol of the order two:

$$(10a) \quad g_{00} = \alpha_2^{-1}(x^1)^{-2} + \alpha_4(x^2, x^3)(x^1)^{-4} + \dots$$

With an analytic metric in the interior of the cylinder — that is for $x^1 < 0$ — the signature becomes $s = -2$ again [10, 6].

In this case we will adjudge topology like the Einstein-Rosen bridge [2, 4]. Then the cylinder with the surface Σ has no interior domain and the co-ordinate domain $x^1 < 0$ has to be identified with the exterior domain $x^1 > 0$. These geometrical peculiarities can be interpreted as a degeneration of the cylinder to a world line with singular metric. $x^1 = 0$ is then the origin of the three-dimensional co-ordinates. The prototype of such solutions is the Schwarzschild-space in the Einstein-Rosen co-ordinates [4]:

$$(11) \quad ds^2 = -4(2m + (x^1)^2)(dx^1)^2 - (2m + (x^1)^2)^2[(dx^2)^2 + \sin^2 x^2(dx^3)^2] + (x^1)^2(2m + (x^1)^2)^{-1}(dx^0)^2.$$

For this choice of the space-time topology we have a static gravitational field without singularities of the field equations and the boundary conditions are fulfilled. This solution represents a particle with the energy E which according to the Einstein-Pauli theorem [5, 11]:

$$(12) \quad 8\pi E = - \oint_{x^1 \rightarrow \infty} \sqrt{-g} g^{00} g_{00,i} n^i ds_2 = 2 \oint_{\Sigma x^0=0} \sqrt{-g} g^{00} \Gamma_{00}^i \delta_i^1 dx^2 dx^3 =$$

$$= - \left[\sqrt{g_{00}} g_{00,1} \oint \gamma g^{11} dx^2 dx^3 \right]_{x^1 \rightarrow 0} = 8\pi m, \quad (c=1) (*)$$

But we are able to attribute an interior domain to the particle without

(*) It is possible to derive from the Einstein-Pauli theorem that for an analytic metric has no interior domain $x^1 < 0$. Then, if the topology of three-dimensional space $x^0=0$ permits to make use of the integral theorem of Gauss there are a contradiction between the realization of the field equations on the interior domain and the boundary conditions.

hypotheses of analyticity. Firstly is

$$(13a) \quad ds^2 = g_{ik} dx^i dx^k + \left[\frac{(x^1)^2}{2m} + \dots \right] (dx^0)^2$$

a static solution of the Einstein equations the definite metric

$$(13b) \quad ds^2 = g_{ik} dx^i dx^k - \left[\frac{(x^1)^2}{2m} + \dots \right] (d\bar{x}^0)^2$$

is a reell solution too [13]. Then (13b) arise from (13a) by the pure imaginary transformation

$$(14) \quad \bar{x}^0 = ix^0.$$

Restricting the transformation [14] to the domain $x^1 < 0$ there is valid:

$$(14a) \quad \bar{x}^0 = ix^0 \quad \text{for } x^1 < 0 \quad \text{and} \quad \bar{x}^0 = x^0 \quad \text{for } x^1 \geq 0$$

and a space-time with the real static metric

$$(15) \quad ds^2 = g_{ik} dx^i dx^k \pm \left[\frac{(x^1)^2}{2m} + \dots \right] (dx^0)^2 \quad \begin{cases} + & \text{for } x^1 \geq 0 \\ - & \text{for } x^1 < 0 \end{cases}$$

results there. On the surface $x^1 = 0$ the metric is continuous with continuous first derivatives therefore the Einstein equations are satisfied on the surface [9, 13].

The metric for $x^1 \geq 0$ may be the Einstein-Rosen metric. In order to make the domain $x^1 < 0$ to an interior of a cylinder, we have to choice the co-ordinates-system in such form that g is vanishing for two values of the co-ordinates. To this end we make the transformation

$$(16a) \quad x^1 = \varrho^2 - 1$$

and we receive

$$(16b) \quad \begin{cases} g_{00} = (\varrho^2 - 1)^2 [2m + (\varrho^2 - 1)^2]^{-1}, \\ g_{\varrho\varrho} = \left(\frac{\partial x^1}{\partial \varrho} \right)^2 g_{11} = -16\varrho^2 [2m + (\varrho^2 - 1)^2], \\ \sqrt{-g_{22}g_{33}} = \sin x^2 [2m + (\varrho^2 - 1)^2]^2. \end{cases}$$

We can see that g and therefore g_{00} are vanished for $\varrho = +1$ and $\varrho = -1$.

So it is possible to postulate that for the co-ordinate domains

$$(17a) \quad \varrho > 1 \quad \text{and} \quad \varrho < -1$$

the Einstein-Rosen metric with Lorentz signature are valid but that in the co-ordinate domains

$$(17b) \quad 1 > \varrho > -1$$

a metric with definite signature is valid.

This definite metric has the peculiarity also that for $\varrho = 0$ the component $g_{\varrho\varrho}$ of the three-dimensional metrical tensor g_{ik} is vanished. If we interpret $\varrho = 0$ like the origin of the three-dimensional co-ordinates then there exists a world-line on the interior of the cylinder on which line the metric has the semi-definite signature $s = -3$. On this line the metric of the three-dimensional space $x^0 = 0$ is singular: $g^{\varrho\varrho}$ has a pole of the order two. With the above-mentioned arguments from earlier papers we are able to prove that on this line the Einstein equations are satisfied too.

We interpret the cylinder (respective three-dimensional: the spheroid) with the surface $\varrho^2 = 1$ like the interior of a particle. The Einstein equations are satisfied on any points. But the metric of the space-time is non-analytic and the three-dimensional contravariants g^{ik} are for $\varrho = 0$ singular. Further is g^{00} for $\varrho = \pm 1$ singular.

This particle possesses according the Einstein-Pauli theorem a rest mass (12). This mass is concentrated in the interior domain on the line $\varrho = 0$. For we want make use of the theorem of Gauss for a three-dimensional space $x^0 = 0$ we have to excluded the neighbourhood of the space point $\varrho = 0$. Therefore with (12) we have:

$$(18) \quad \left\{ \begin{aligned} 8\pi E &= - \left[\sqrt{g_{00}} \frac{\partial g_{00}}{\partial \varrho} \oint \gamma g^{\varrho\varrho} dx^2 dx^3 \right]_{\varrho^2 \rightarrow 1} \\ &= \oint \sin x^2 dx^2 dx^3 \left[\sqrt{g^{00}} \frac{\partial g_{00}}{\partial \varrho} \sqrt{|g_{\varrho\varrho}|} (2m + (\varrho^2 - 1)^2) \right]_{\varrho \rightarrow 0} \\ &= 4\pi \cdot 2m. \end{aligned} \right.$$

The non-analytic metric we are treated above may be the most simple model of an elementary particle in general relativity. This existence of a definite metric on the three-dimensional limited domain is very important for this model. The existence of a world line on which the contravariant metric of the three-dimensional space is singular is a necessary condition for the existence of this domain. Simultaneously this particle model is the

most simple continuation of the Schwarzschild metric for which continuation the Einstein vacuum equations are satisfied on each points. In comparison with the Einstein-Rosen space-time we may call our space-time the non-analytic continuation of Schwarzschild space-time without bridge.

Of course our model has few physical properties. There it do'nt give conditions for the proper radius and the mass of the particle. But that is'nt very astonishing: the particles with rest mass have not only gravitational properties but also the particles are coupled with many non-gravitational fields (electromagnetic fields, nuclear fields etc.). Therefore we can hope that the consideration of the matter fields and consequently the change-over from the Einstein empty space equations to the coupled equations of gravitational fields and of matter fields are giving general relativistic model of particles which possess a richer structure. Indeed, Stephenson [8] has noted recently that the introduction of a space time with variable signature leads to a mass spectrum for Fermi particles in the special relativistic approach yet.

But the pure gravitational particle models give a hint for the discussion of the problem of gravitons. Of course, the gravitons have only pure gravitational and mechanic properties. Therefore we must be able to describe the gravitons in the Einstein theory of empty space. Therefore the discussion of pure gravitational models of elementary particles shall be a first step for the solution of the problem of gravitons.

While our model is corresponding with a particle with rest system the gravitons do'nt have rest systems. In the space-time diagram it's meaning that the cylinder $\varrho^2 \leq 1$ respective the world line $\varrho = 0$ of the cylinder centrum must be turn round the angle $\pi/4$. In the same way we have to modify the boundary conditions. But the hypersurface of the new cylinder does'nt be isotrop. There are no cylindrical hypersurface which are isotropic on each points. As we have noted in earlier paper we should take the gravitons as point-like particles, it means the cylinder would degenerate into a world line. Then, in this model the graviton is represented by a singular world line which belongs to a congruence of geodesic world line which otherwise is isotropic.

The determinant g vanishes on this singular geodesic. In this case the matrix $g_{\nu\nu}$ has not the rank 3 but it has the rank 2. Then in this case the surface and the centrum of the cylinder and therefore both singularities of the contravariant $g^{\nu\nu}$ are coinciding. Therefore the inertial index is on the singular geodesic: $(0, 0, -1, -1)$.

We remark that the conception of pointlike gravitons has a clear physical meaning. We must have 3 independent universal constants to the construction of an elementary length. The three constants wich are at our disposal from the quantum theory of gravitational field are: the velocity of light c , Planck's constant $\hbar$, the gravitational constant f .

But the gravitational constant f is describing the interaction of the gravitational field with the matter fields. The constant f doesn't appear in the theory of free gravitational field. Therefore, it is not possible to attribute proper radii to the free gravitons. But the gravitational constant f appears if the gravitons interact with matter fields. And the existence of Planck's elementary length which is resulted from the quantum theory of gravitational field will be efficient.

I think that the decision about the existence or non-existence of general relativistic particle models like gravitons is the criterium to reply the question whether Einstein's idea of pure geometric field theory is physically hopeful or not.

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INTERVENTI E DISCUSSIONI

— A. KOMAR:

1) It is not necessary to change the signature of the metric to obtain a Ricci-flat topologically Euclidean continuation of the Schwarzschild metric across the Schwarzschild radius.

2) If we take seriously the internal symmetric group SU_3 we are led to consider an Einstein space with positive-definite signature first given by Vranceanu and Egoroff. However since that manifold is homogeneous it does not appear natural to select a boundary across which we should piece it to a manifold with Lorentzian signature. It is of course possible to do so via the methods of this paper, but I suspect that the relation between the internal space and the external space is more subtle.

— H. J. TREDER:

1) I agree that other empty-space-continuation of the Schwarzschild

metric across the Schwarzschild surface are existent. My point is that a continuation exists for which the Ricci-flat interior has a definite signature. My space is a static space in the interior too.

2) My model is a very simple model to prove the possibility of a definite Ricci-flat metric in the interior. The symmetries are the symmetries given by the Lie-group for a definite Schwarzschild space I have a spherical symmetric particle.

— J. A. WHEELER:

It would be of interest to all of us to know if Professor Lichnerowicz agrees with the conclusion of Professor Treder, that the signature can change from $-+++$ to $++++$ across a hypersurface, and this without the curvature becoming infinite as the hypersurface is approached.

This kind of behaviour would seem to clash in the most direct way with the equivalence principle as one has been accustomed to understand that principle in the past. Given any point anywhere in spacetime, one normally says; then one can introduce at this point a Riemann normal co-ordinate system; and the first term in this power series of Riemann represents the standard Lorentz geometry. In this or any other customary formulation of the equivalence principle, the local physics at any point is identical in character to the local physics at any other point. This identity in character holds—according to one's usual understanding of the principle—no matter how strong the gravitational field may be. The strength of the gravitational field shows up only in tide-producing forces of the components of the Riemann curvature tensor or the higher order terms in Riemann's standard power series development for the metric. The strength of the gravitational field has no influence whatsoever on the first or leading term in Riemann's power series. That term always and everywhere has the standard Lorentz form.

In other words, at every point a sudden explosion will release a light pulse, and this light pulse will travel on a light cone of standard character. Never does a light pulse crash into a barrier. But a region where the metric has the character $++++$ cannot support the propagation of a light pulse.

It is a barrier against light. The presence of such a barrier would seem to stand in the most direct contradiction with everything one normally understands by the equivalence principle.

— D. IVANENKO:

I should like to ask Professor Wheeler if the fluctuations in the metric which he has discussed in his book do not open up the possibility to have a signature at some places of the form $+++$.

— J. A. WHEELER:

So far as I can see one speaks in the quantum theory of general relativity of the probability amplitude $\psi^{(1)}(\mathcal{G})$ to have this, that or the other spacelike *three*-geometry (signature $+++$); there is no place for the signature of the 4-geometry to make an approbance.

[Alla discussione parteciparono anche A. PAPAPETROU, A. LICHTNEROWICZ, J. SCHARE, D. IVANENKO.]

Vérification Expérimentale de la Théorie de l'Effet Inertial de Spin (E.I.S.).

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Une théorie a été proposée [1] pour le T_M non symétrique en relation avec la présence du spin. Cette théorie postule la non colinéarité de la vitesse avec l'impulsion dans les milieux doués de spin et, corrélativement, l'existence d'un effet de recul en translation d'un corps d'épreuve (C.E.) convenable lors de l'acquisition par celui-ci d'une densité de spin σ^* (soit aussi d'une aimantation spécifique $\sigma = (e/m_0)\sigma^*$).

Un montage original a été mis au point pour éprouver la valeur physique de ces hypothèses et l'expérience, faite les 11-12, 12-13 et 15-16 Août 1964, a effectivement mis en évidence sans ambiguïté l'effet de recul du C.E. lorsque son aimantation spécifique passe d'une valeur $+\sigma$ à une valeur $-\sigma$.

Ce passage de l'état $+$ à l'état $-$ est en fait effectué avec une fréquence de récurrence égale à la fréquence propre ν_0 d'une suspension à très faibles pertes et l'on observe interférométriquement l'amplitude $\pm A$ atteinte par le C.E. ($2A \simeq 0,7\mu$).

Une grande attention a été portée à l'élimination des artefacts par le choix des matériaux dia- ou para-magnétiques, le montage strictement coaxial de l'ensemble; la forme et la durée ($70\mu s$) des impulsions de courant commandant le magnétisme du C.E., les caractéristiques mécaniques de la suspension en quartz excluant, sur sa fréquence propre fondamentale $\nu_0 = 12,7291$ Hz tout mouvement autre qu'une translation du C.E. suivant la direction prévue, ... etc.

Brièvement, l'expérience montre que le C.E. est mis en oscillation permanente lorsque son magnétisme est inversé à la fréquence propre ν_0 de l'équipage ou aux fréquences $\nu_0/(2k+1)$, k entier, qu'il reste immobile au contraire soit quand son magnétisme est inversé aux fréquences $\nu_0/2k$, soit quand il est attaqué par des impulsions de courant toutes de même signe quelle que soit alors leur fréquence de récurrence.

Les effets ainsi observés, complètement inexplicables par la mécanique ou l'électromagnétisme classiques, sont parfaitement conformes

aux prévisions de la théorie de l'E.I.S., y compris l'ordre de grandeur des amplitudes atteintes.

Nous pensons que l'effet est dû aux électrons libres du C.E. actuellement en alliage de Weiss [2] de résistivité $0,30 \Omega \text{ mm}^2 \text{ m}^{-1}$ et qu'un perfectionnement de la théorie [1] est nécessaire pour tenir compte de l'état du nuage électronique.

L'expérimentation se poursuit et va être entreprise sur des ferrites de haute résistivité.

Une note aux *Comptes-rendus* de l'Académie des Sciences est sous presse [3] et un compte-rendu expérimental et théorique détaillé de ces travaux sera publié prochainement.

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Problems and Prospects in Quantization of Relativity. (*)

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1. - Introduction.

In this report, we shall discuss some current aspects of the quantization problem in general relativity, together with closely related classical questions. The viewpoint is that of Lorentz covariant quantum field theory, as employed in the joint work with R. Arnowitt and C. W. Misner [1] on which most of this discussion will be based.

2. - Classical theory.

The close correspondence which has been found between general relativity and usual field theory holds only if one keeps the same physical boundary conditions in the former, as one always does in the latter. This means restricting oneself to solutions (or states) for which (1) space is asymptotically flat, so that there exists a frame in which the metric $g_{\mu\nu}$ approaches its Minkowskian form $\eta_{\mu\nu}$, and its derivatives vanish, as $1/r$, and (2) no topological complications occur, and there is Euclidean connectedness. Physically, these restrictions have quite intuitive significance (*e.g.* elimination of cosmological solutions): we are considering only bounded systems with finite energy content, *i.e.* states imbedded in an asymptotically flat space-time. The specific rate of approach to flatness at spatial infinity is closely related to boundedness, as any slower rate would imply a divergent energy content.

Both the analogy to all other physical systems and the existence of a Newtonian correspondence limit of general relativity actually require the $1/r$ asymptotic behavior. The consistency of these boundary conditions is non-trivial here, owing to the initial value (constraint) equa-

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tions. One may show, however, that there exists a « wave-front » theorem [2] which states that $1/r$ metric behavior is possible if and only if the dynamical modes of the metric field (discussed below) fall off much faster than $1/r$.

The field-theoretical analysis of the classical gravitational field is complete [1] and we sketch briefly the physical features brought out by the canonical Hamiltonian formalism. The general coordinate invariance of the theory means simply that it has the same form as the « parameterized » formulation of ordinary dynamical systems, in which the coordinates are treated on an equal footing with the other variables. In both cases, the dynamics emerges only after one chooses a specific (but arbitrary) coordinate frame. Once this is done, and the gauge parts of the field are isolated, the redundant constraint variables of the theory (analogous to the longitudinal electric field in Maxwell theory) can be, in principle, eliminated. As a result, one is led to a canonical form for the action or Lagrangian, in terms of two polarization degrees of freedom (as befits a self-interacting spin 2 massless system.) This action reads (in a frame analogous to the radiation gauge in electrodynamics),

$$(1) \quad I = \int L d^4x = \int d^4x [\pi^{ijTT} \partial_0 g_{ij}^{TT} - \mathcal{H}(\pi^{TT}, g^{TT})]$$

where g_{ij} are the spatial covariant components of $g_{\mu\nu}$, the π^{ij} are essentially the coefficients of the second fundamental form, and the symbol « TT » denotes the normal transverse-traceless part of the g_{ij} and π^{ij} considered as symmetric arrays (thus, $\pi^{ijTT}{}_{,j} = 0 = \pi^{ijTT}$). The Hamiltonian, $\int \mathcal{H}(\pi^{TT}, g^{TT}) d^3x$, is the generator of time translation in the chosen frame. It cannot be given in closed form in terms of the four conjugate components π^{TT}, g^{TT} owing to the complicated self-interaction of the field. However, it is perfectly well-defined as a power series, say. Its numerical value in a given state is just the energy of the system. The corresponding spatial momentum $\mathbf{P}$, by contrast, is of a very simple quadratic form in the « TT » variables, being identical to the linearized theory's $\mathbf{P}$. In addition to the energy and momentum, the form (1) yields also the Poisson Bracket relations among the two degrees of freedom:

$$(2) \quad [\pi^{lmTT}(\mathbf{r}), g_{ij}^{TT}(\mathbf{r}')]_{PB} = \delta_{ij}^{lm}(\mathbf{r} - \mathbf{r}')$$

where $\delta_{ij}^{lm}(\mathbf{r})$ is the Dirac delta function in the space of « TT » functions. It may also be noted that vanishing of these degrees of freedom at a given time implies, as is physically desirable, that space is flat for all time (vacuum state). Before going on to quantization on the basis of this complete Hamiltonian theory, let me come to some general classical

results and problems concerning the energy-momentum P_μ . Classically, the canonical approach leads to a definition of P_μ which (1) agrees with that of other systems coupled to general relativity and with the Newtonian limit, (2) is conserved, and (3) transforms precisely as is physically required under co-ordinate transformations [3]: we have that P_μ behaves as a free Lorentz four-vector under co-ordinate transformations of the form $x' = (Lx)^\mu + \xi^\mu(x)$ where L is a rigid Lorentz rotation and ξ^μ are four arbitrary functions such that $\xi^\mu_{,\nu} \sim 1/r$ asymptotically. Incidentally, as might be expected, P_μ is finite only if the «TT» components fall off faster than $1/r^3$, just as is the case for dynamical modes of all other fields. The energy question in the classical theory thus seems to us settled entirely satisfactorily (with correspondence to the usual definitions which assume the stronger condition that $\partial_\alpha g_{\mu\nu} \sim 1/r^2$ rather than merely $1/r$), both as to definition, physical significance and transformation properties. Now let me mention two essential unsolved problems. As is well known, a quantized field is physically reasonable only if (a) it has a vacuum or lowest energy state, so that $P_0 \geq 0$ for all states. (b) All states have timelike (or possibly null) P_μ . This means, in terms of the invariant mass m , that $P_0^2 - \mathbf{P}^2 = m^2 \geq 0$, so that the mass is real. Both these properties hold for, say, the free Maxwell field and, less trivially, for the linearized theory, but neither has been established generally [4] for the free classical Einstein field. Point (a) is especially important here as it can be shown that existence of any negative energy states implies no lower bound to the energy spectrum. Decisive results, even classically, on these points would be very important for the quantization program. Finally, it should be noted that coupling to matter presents no new problems and goes through straightforwardly (even in the case of spinor fields requiring the use of vierbeins [5]), to a simple canonical form.

3. - Quantum theory.

The transition from classical to quantum terms may be discussed at various levels of rigor. In terms of simple «correspondence principle» quantization, all that is needed is to replace the P.B. in eq. (2) by a commutator and thereby achieve quantization, if not consistency. The hard problems, involved in a consistent quantum formulation, have their counterpart in other gauge theories, *e.g.* the coupled Maxwell or Yang-Mills fields. They are related to operator ordering questions, constraint equations, Lorentz invariance and q -number gauge transformations.

To be sure, one has formally, in eq. (1) a system of two degrees of freedom, with perfectly normal c -number co-ordinates x^μ , and a Hamil-

tonian depending only on the free excitations g_{ij}^{TT}, π^{ijTT} . However, the whole theory involves also the remaining (constraint) parts of the metric as well. Thus one may ask whether a (classically) space-like surface remains space-like, *i.e.* whether the eigenvalues of the operator g_{ij} are positive-definite. This is an open question (it is true to lowest order [6]); but in any case, many classical properties of Riemann space will of course disappear when the variables become operators and hold only as expectation values. Naturally, whenever one quantizes a system, many of its basic classical properties vanish; it is perhaps a little more shocking a priori when these notions were of a geometrical nature, but such questions are not so disturbing as the following ones.

To return to the problems of a quantization program, let us talk in terms of a quantization with positive-definite Hilbert space metric, *i.e.* « radiation gauge »—like frames. [Even in electrodynamics such frames seem to be preferred for quantization]. First is the question of consistency between constraint equations, equal-time commutation relations and equations of motion. Classically, this is guaranteed by the Bianchi identities. In operator form, these are so far from being identities, that even in the simpler Yang-Mills system [7], one needs to add nonclassical terms of double commutator form with $\hbar^2$ coefficients to the action to reestablish them (irrespective of the operator ordering used in the constraint variables themselves). The second problem, Lorentz invariance, is perhaps the most fundamental. Classically, there is no problem, as the theory is manifestly covariant. In operator terms, however, one must establish that the ten group generators, P_μ and $J_{\mu\nu}$, (or the corresponding stress-tensor components [8]), have the correct commutation relations by virtue of the fundamental relations among the canonical variables. [Alternately, Lorentz covariance may be checked directly on the field equations or Lagrangian; the problem is then that there is a concomitant gauge (co-ordinate) transformation which is in general a q -number, as in electrodynamics, and highly non-trivial.] The difficulty that arises in verifying the $P_\mu, J_{\mu\nu}$ relations is due to the implicit nature of P_0 . As was noted long ago by Dirac, the usual form of Hamiltonian dynamics makes the spatial generators P_i, J_{kl} very simple in form, namely purely kinematical quadratic functionals of the fields, but makes up for this in complicating the time parts P_0, J_{0i} , *i.e.* the energy density T_0^0 . This makes the basic problem fall on the time translation and Lorentz rotation parts. Recent attempts by Dirac [9] and Schwinger [10] have tried to circumvent this difficulty by formally keeping the gauge variables in the theory, with standard commutation relations but without quite raising them to the level of the true dynamical operators. The gauges then enter as a sort of parameter. In this way one works, not with the implicit solutions T_μ^0 of the constraint equations, but with the equations themselves. A choice

of coordinate is then only imposed at the end. Whether this method is fully equivalent to the more difficult procedure seems to be an open question still; it may well have to face the same problems in another way. This is made likely by the fact that quantization in different gauges (coordinate frames) within the « radiation gauge » class may well lead to different results if these differ from each other by q -number transformations. This third major question has to do with the fact that while the co-ordinates in any one frame are of course c -numbers, the relations between two sets may involve operators. In that case, it is not known whether (a) quantization in all frames is consistent and (b) even where it is consistent, whether the resulting quantum theories are equivalent. There is no a priori reason to believe the latter possibility, at the quantum level, just because there is classical equivalence. Indeed there exists evidence [11] that this is not the case for some very simple models.

Having noted some of the conceptual and technical problems in the way of a resolution of the consistency of quantization (problems which have their counterparts in simpler theories also, as we have noted), we should end by pointing out that most interesting « practical » questions can certainly be investigated at the « correspondence quantization » level without worrying about such rigor problems. Indeed, as interesting and unexpected results arise from « naive » quantum considerations, they will certainly stimulate further work on the compatibility between these two pillars, the one macroscopic and the other microscopic, of modern physics.

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Parole di commiato

pronunciate

da M.me M. A. Tonnelat

a nome di tutti i partecipanti al Convegno.

Adesso, vorrei dire qualche parola per tutti i partecipanti alle giornate galileane. Forse, il fatto di scegliere un'ambasciatrice è come la garanzia d'una rappresentanza internazionale.

Depuis quelques jours et grâce à vous, Monsieur Sestini, qui vous êtes tant dépensé pour assurer le Secrétariat de ce Colloque, grâce à vous aussi, Monsieur Cattaneo, nous avons été comblés et tout fut parfait: Le soleil couchant était au rendez-vous à Arcetri, et, hier soir, cette resurrection de la musique de la Renaissance a été une grande joie pour beaucoup d'entre nous. Certains, et moi même, aurions quelquefois aimé être des « dames » pour profiter plus à loisir des entrées permanentes dans les musées...

Hélas! On ne peut réunir dans un temps aussi bref tous les centres d'intérêt: Florence et le Symposium en offraient de trop nombreux!

J'arrive donc au but plus précisément scientifique de ces journées relativistes galiléennes.

Elles nous ont donné une impression de progrès considerable au cours de discussions particulièrement euphoriques et cordiales. Aidés par la possession de documents qu'une organisation aussi magistrale que vigilante nous avait procurés, il a été possible de formuler explicitement des questions traditionnellement obscures ou même épineuses par une sorte de vocation: la signification des tétrades, la définition d'une énergie gravitationnelle dans un espace courbe au dans un espace plat, le rôle et la portée des explications spinorielles, les critères de la notion d'observable en Relativité générale.

Seule, la modestie Italienne, nous a privés de contributions dont nous connaissons néanmoins la valeur et que nous savons apprécier. Elles figurent virtuellement, enrichies de leur courtoisie, dans cet hommage à Galilée.

J'emporte pour ma part une très reconfortante impression de progrès qu'a permis une atmosphère ouverte et comprehensive, celle même qui requéraient si passionnément Galilée et Jean XXIII, celle qui doit caractériser l'esprit scientifique.

Pour tout cela je voudrais vous remercier, Messieurs; non seulement vous-même mais aussi les absents: Monsieur La Pira, Maire de Florence qui nous a si chaleureusement et, je dirais, si « personnellement » accueillis au premier jour de ces Réunions; Monsieur Polvani, Président du Comité National, les membres du Comité d'organisation, le Secrétariat si dévoué et, tout spécialement, Madame Cattaneo.

Pour terminer, je voudrais insister sur le caractère spécifique de ces Journées organisées sous l'égide de Galilée et placées, en somme, à l'ombre de son souvenir ou plutôt à la lumière de son oeuvre.

Permettez-moi de parler un instant en historien des Sciences dont le Symposium est si proche.

La principale vertu des théories passées — que certains croient mortes, — tient essentiellement pour nous, Physiciens, dans leur caractère stimulant. Si dépassés que soient leurs concepts, — et il faut bien avouer qu'ils le sont, — ces théories nous lèguent leur style, leur ambition. Sans elles notre effort n'aurait aucune racine et, pour cette raison, presque aucun sens.

Si la Science grecque fut la première à concevoir cette idée extraordinaire que la physique pouvait être non seulement un moyen d'utiliser l'expérience mais qu'elle devait, par surcroît, la comprendre, c'est Galilée et la Renaissance qui apportent néanmoins une réponse décisive à cette pseudo-Relativité de Protagoras: « L'homme est la mesure de toute chose ».

A cette assertion, Galilée oppose le germe d'un authentique Principe de Relativité. L'homme ne peut s'éprouver lui-même que dans un dialogue avec l'expérience et la mesure de ses jugements réside dans leur cohérence. Au *Fatum* du destin va se substituer bientôt l'absolu de la loi.

Peut-être ce rejet d'un absolu propre à l'homme lui fut-il dicté par la mesure de vos monuments, par la pureté de vos paysages. L'homme s'intègre plus volontiers à un monde construit suivant ses harmoniques.

La naissance d'une Physique Mathématique liant expérience et Principes pour aboutir à ces « Observables de la Relativité générale » n'est-elle pas l'expression technique d'un humanisme dépouillé maintenant de ses attaches sensibles? Si elle ne l'est pas toute la physique, fût-elle relativiste, vaut-elle seulement une heure de dépense? Est-il donc étonnant que ses débuts aient été associés à la clarté de ce Pays dont le romantisme de Goethe devait conserver la nostalgie?

Peut être ainsi, pour la même raison, ces Journées qui ont permis de percevoir l'accord secret des mathématiques et de l'expérience, de

Lichnerowicz et de MacVittie, ont-elles bénéficié de cette contemplation que nous proposait le Maire de Florence au premier jour de ce colloque.

Monsieur Sestini et vous aussi Monsieur Cattaneo, vous avez permis ainsi une Réunion unique, exceptionnelle et, si ce n'était manquer aux exigences de la Relativité, je dirais privilégiée.

C'est donc l'expression de notre reconnaissance tout aussi exceptionnellement chaleureuse que je voudrais vous transmettre et que j'ai essayé, bien maladroitement, de vous traduire ici.

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